STATE ESTIMATION FOR LINEAR SYSTEM WITH JUMP PARAMETER AND ITS APPLICATION TO CONTROL

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Abstract

In this dissertation, the state estimation problems for linear (1) discrete, (2) continuous, and (3) continuous-discrete systems with jump parameters are considered; and the optimal and the approximate estimator algorithms are presented for those jump-parameter systems and applied to the optimal control problems.

First, we consider the estimation problem for a linear discrete system with a Markov chain, where the system is described by a stochastic difference equation modulated by a Markov chain. The optimal estimator algorithm is derived by use of Bayes' rule. The optimal algorithm, however, requires the ever-growing amount of memory, so that a feasible approximate estimator algorithm is proposed. An adaptive sequential estimator algorithm is also presented for the case when the transition probabilities of the Markov chain are unknown but fixed. Furthermore, the obtained estimator algorithms are applied to the special discrete systems, that is, linear discrete systems with interrupted observations.

Secondly, we treate the estimation problem for a linear continuous system with a jump process, where the system is represented by a stochastic differential equation and modulated by a jump Markov process. The optimal estimator algorithm is derived. The crucial points for the derivation of the optimal algorithm are

(1) to express the jump process in terms of its initial value, the jump times and the values taken after the jumps, instead of its instantaneous values, and (2) to apply general Bayes' rule to obtain the a posteriori probability distribution of the jump process. The optimal estimator algorithm is, however, infinite-dimensional, so that a feasible finite-dimensional algorithm is also proposed.

Thirdly, we consider the estimation problem for a linear continuous-discrete system with a jump process, where the system is described by a stochastic differential equation, while the observations are made at discrete times; and both the system and the measurement subsystem are modulated by a jump process. The optimal estimator algorithm and its approximate one are presented by the same approach as adopted for continuous systems.

Finally, we consider the optimal control problems for (1) a linear discrete system with switching environments, (2) a linear continuous-discrete system with switching environments, and (3) a linear discrete system modulated by a Markov chain. The optimal control algorithms are derived for the first two systems; and for the third system, a suboptimal control algorithm is proposed. The optimal and the suboptimal control algorithms are coupled with the estimator algorithms presented above.

CONTENTS

CHAPTER I

INTRODUCTION

1.1 Introduction and Historical Remarks

In the real world, almost everything accompanies uncertainty, so that its future behavior can seldom be predicted without errors. Therefore, one of the problems facing the engineer, and much of mankind, in making decisions is how to intelligently utilize information from observations available. Usually, we should thus predict the uncertain situation or state based upon the available information and make a decision so as to minimize, in some sense, the difference between the predicted state and the desirable one which we have in mind. Consequently, the estimation or prediction is the subject of great importance in practical situations; because it provides us with the foundation for making decisions in uncertain environments.

The estimation problems have been considered from the earliest times, and the first rigorous studies of least-squares estimation in stochastic processes were made by Kolmogorov (1939) and Wiener (1949), where the work of Wiener was independent of Kolmogorov's and their aimes were somewhat different; Kolmogorov treated the prediction problem for discrete-time stationary processes, while Wiener considered the continuous-time linear prediction problem for stationary processes stressing the engineering significance of his ideas and results. Several studies on generalization and improvement of works of Kolmogorov and Wiener were made thereafter. A remarkable result was obtained by Kalman (1960) who changed the conventional formulation of the prediction problem by assuming a "Markovian model" as the output of a dynamical linear system driven by white noise. Specifically, he assumed that the output process $y(k)$ could be represented by

$$x(k+1) = F(k)x(k) + G(k)u(k) \tag{1.1}$$

$$y(k) = H(k)x(k), \tag{1.2}$$

where $x(k)$ is an $n \times 1$ state vector and $u(k)$ is an $m \times 1$ independent Gaussian zero-mean noise process with covariance matrix $Q(k)$. It is assumed that the initial state $x(0)$ is Gaussian with mean $\hat{x}(0)$ and covariance $\hat{P}(0)$ and is independent of $u(k)$. Also, vector $\hat{x}(0)$ and matrices $F(k)$, $G(k)$, $H(k)$ and $\hat{P}(0)$ are assumed completely known. Under these assumptions, by introducing a notion of innovations, Kalman solved the linear least-squres prediction problem in a recursive form, which is significantly suitable for computer-aided practical implementation.

In many practical situations, however, matrices $F(k)$, $G(k)$ and $H(k)$ in (1.1) and (1.2) are not completely known as deterministic matrix-functions of time k, because there may exist identification errors of these matrices and random changes of the components of these matrices. For systems having such partially known matrices, Magil (1965) made the first contribution in the estimation problem, where considered was a scalar discrete system with unknown constant parameters. The adaptive estimation problem treated by Magil was extended to the vector case by Hilborn and Lainiotis (1969). Also, a closely related problem of detection of the presence or absence of a signal was considered by Middleton and Esposito (1968).

Meanwhile, Nahi (1969) considered the optimal linear estimation problem for a linear discrete system with uncertain observations described as follows:

$$x(k+1) = F(k)x(k) + G(k)u(k) \tag{1.3}$$

$$y(k) = \gamma(k)H(k)x(k) + v(k), \tag{1.4}$$

where matrices $F(k)$, $G(k)$ and $H(k)$ are known; $u(k)$ and $v(k)$ are zero-mean mutually independent Gaussian white noises with known covariance matrices; and $\gamma(k)$ is an independent sequence or an unknown constant taking on values of 0 or 1. Nahi derived the optimal linear estimator algorithm for system (1.3) and (1.4) and made the first contribution in the estimation problem for systems with "switching" parameters. Ackerson and Fu (1970) proposed a suboptimal estimator algorithm for linear discrete systems with switching environments, where characteristics of the noises change according to the values of the switching parameter. The estimation problem for system (1.3) and (1.4) with nonswitching $\gamma(k)$ was also examined by Dajani and Sage (1970). For the case when $\gamma(k)$ is a Markov chain, the optimal nonlinear estimator was derived by Jaffer and Gupta (1971); and independently, Sawaragi, Katayama and Fujishige (1972) presented a suboptimal sequential estimator algorithm, because the optimal algorithm requires the evergrowing amount of memory. Also, Sawaragi, Katayama and Fujishige (1973) considered an adaptive estimation problem where the transition probabilities of the Markov chain were unknown but fixed.

The above-mentioned works were, however, mostly concerned with discrete systems. Lainiotis (1971) first gave the complete solution of the estimation problem for continuous systems with nonswitching parameters. Also, the multi-shot joint detection and estimation was examined in (Lainiotis, 1972), where it was assumed that the unknown parameter was constant in each specified signaling interval and that the switchings of the parameter could occur at the ends of the signaling intervals only. For the case when the uncertain parameter is expressed in terms of a jump Markov process, Sawaragi, Katayama and Fujishige (1974), and Fujishige and Sawaragi (1974 a) derived the optimal estimator algorithm and proposed its feasible approximate estimator algorithms. Also, the estimation problem for continuous-discrete systems with jump parameters was examined in (Fujishige and Sawaragi, 1974 b).

Optimal control problems for system (1.3) and (1.4) with a control input term in (1.3) have been considered from earlier times; Bellman and Kalaba (1961) formulated, as the interrupted stochastic control process, the problem of determining the optimal control policy in a situation where at any time there is a non-zero probability that the state cannot be observed. Such an optimal control problem was considered by Eaton (1962), Fujita and Fukao (1973), and Lainiotis, Upadhyay and Deshpande (1973); and they asserted that usual "Separation Theorem" did hold. However, Fujishige (1974) pointed out that their results were incorrect and that "Separation Theorem" did not hold. Similarly, the continuous counterpart treated by Lainiotis, Deshpande and Upadhyay (1972) was incorrect. Recently, the estimation and control problem for uncertain linear continuous-discrete systems has been examined by Lee and Sims (1974). Related control problems for continuous systems with jump parameters were considered in (Sworder, 1969; Wonham, 1970) where the complete observations of both the jump parameters and the state were assumed.

Also, other related problems of estimation of jump processes have been considered in (Wonham, 1965; Shiryaev, 1966; Yashin, 1968).

1.2 Outline of the Dissertation

In chapter II, we consider the estimation problem for linear discrete systems with a Markov chain. The system to be considered is represented by a stochastic difference equation modulated by a Markov chain. The estimation problem is formulated in section 2.2. The optimal estimator algorithm is derived in section 2.3; and in section 2.4 the optimal estimator algorithm is also presented in another form, which is useful for deriving the optimal and the approximate estimator algorithms for continuous systems with jump parameters. The optimal estimator algorithms, however, require the ever-growing amount of memory, so that feasible approximate estimator algorithms are proposed in section 2.5. In section 2.6, treated is an adaptive estimation problem for the case when the transition probabilities of the Markov chain are unknown but fixed. The optimal and the approximate estimator algorithms obtained in the previous sections are applied, in section 2.7, to linear discrete systems with interrupted observations. In section 2.8, simulation studies are carried out to demonstrate the feasibility of the proposed approximate estimators.

In chapter III, we consider the estimation problem for linear continuous systems with jump processes. The system to be considered is described by a stochastic differential equation and modulated by a jump Markov process. The estimation problem is precisely formulated in section 3.2. Section 3.3 is devoted to the derivation of the optimal estimator algorithm. The approach adopted is 1) to express the jump Markov process in terms of the initial value, the jump times and the values taken after the jumps, and 2) to apply general Bayes' rule (Kallianpur and Striebel, 1968) or Lainiotis' formula (Lainiotis, 1971) to obtain the a posteriori probability distribution of the jump process. The minimum-variance estimate is given in terms of the a posteriori probability distribution of the jump process and the Kalman-filter estimates corresponding to the admissible values of the jump process. The optimal estimator algorithm is, however, infinite dimensional, so that feasible approximate estimator algorithms are proposed in section 3.5. Simulation studies are also carried out, in section 3.6, to illustrate the behavior of the optimal estimator and to demonstrate the feasibility of the proposed approximate estimator algorithms.

In chapter IV, we extend the results obtained in the previous chapters II and III to linear continuous-discrete systems with jump parameters. That is, the system is represented by a stochastic differential equation, while the observations are made at discrete times; and both the system and the measurement subsystem are modulated by a jump process. The estimation problem is formulated in section 4.2 and the optimal estimator algorithm is presented in section 4.3. For the same reason as in continuous systems treated in chapter III, the optimal estimator algorithm is not feasible, so that a feasible approximate estimator algorithm is proposed in section 4.4. In section 4.5, simulation studies are carried out to show the feasibility of the proposed approximate estimator algorithm.

Chapter V is devoted to the optimal control problems for stochastic linear systems with jump parameters. Considered are the following three cases: 1) the optimal control problem for linear discrete systems with switching environments, in section 5.2; 2) the optimal control problem for linear continuous-discrete systems with switching environments, in section 5.3; and 3) a suboptimal control problem for linear discrete systems with a

Markov chain, in section 5.4. For all the control problems treated here, we take an expected quadratic cost as a performance criterion. A certainty equivalence is shown to hold for the control problems treated in sections 5.2 and 5.3 for linear discrete and continuous-discrete systems with switching environments. The optimal control input is given interms of 1) the optimal control input for the usual linear-quadratic-Gaussian systems and 2) its correction term.

In completing the dissertation, the already published and submitted papers laid its foundations as follows: Sawaragi, Katayama and Fujishige (1972), and Sawaragi, Katayama and Fujishige (1973) for chapter II; Sawaragi, Katayama and Fujishige (1974), and Fujishige and Sawaragi (1974 a) for chapter III; Fujishige and Sawaragi (1974 b) for chapter IV; and Fujishige and Sawaragi (1974 b) and Fujishige and Sawaragi (submitted 1974) for chapter V.

CHAPTER II

STATE ESTIMATION FOR LINEAR DISCRETE SYSTEM WITH MARKOV CHAIN

2.1 Introduction

In this chapter, we consider the state estimation problem of linear discrete systems with a Markov chain; that is, the system to be considered is described by stochastic difference equations and is modulated by a Markov chain. Such a problem arises in reliability theory context where there is a possibility of system-component failure; or in communication theory context where the signal process is subjected to random attenuation or fading.

The estimation problem is precisely formulated in section 2.2. Section 2.3 is devoted to the derivation of the minimum-variance estimator algorithm. The optimal estimator algorithm is presented in section 2.4 in another form, which is useful for deriving estimator algorithms for continuous systems (see Chapter III). The optimal estimator algorithms derived in sections 2.3 and 2.4 are, however, practically infeasible because the evergrowing amount of memory is required. Therefore, for practical implementation, we propose in section 2.5 feasible approximate estimator algorithms corresponding to the optimal algorithms presented in sections 2.3 and 2.4. We further consider an adaptive estimation problem, where the transition probabilities of the Markov chain are unknown but fixed. A sequential adaptive estimator algorithm is presented in section 2.6. In section 2.7, considered is the estimation problem for systems including the Markov chain in a special way; that is, linear systems with interrupted observation mechanisms, where if the Markov chain is equal to 1, the observation contains the information concerning the state, while if the Markov chain is equal to 0, the observation consists of noise only. The approximate estimator algorithms proposed in sections 2.5 and 2.6 are rewritten for the linear discrete systems with interrupted observation mechanisms. In order to demonstrate the feasibility of the proposed approximate estimators, digital simulation studies are carried out, in section 2.8, for the special system treated in section 2.7.

2.2 Statement of Problem

Consider the system represented by a linear stochastic difference equation

$$x(k+1) = F(k,\gamma(k+1))x(k) + Q(k,\gamma(k+1))w(k) \qquad (2.1)$$

$$k = 0, 1, \ldots,$$

and let the observation be given by

$$y(k) = H(k,\gamma(k))x(k) + R(k,\gamma(k))v(k) \qquad (2.2)$$
$$k = 0, 1, \ldots,$$

where

$x(k)$: an $n \times 1$ state vector at the kth sample time;

$y(k)$: a $p \times 1$ observation vector;

$w(k)$: an $m \times 1$ Gaussian white noise vector with zero mean and unit variance matrix;

$v(k)$: a $p \times 1$ Gaussian white noise vector with zero mean and unit variance matrix;

$\gamma(k)$: a Markov chain taking on values of $1, 2, \ldots, M$;

$F(\cdot,\cdot)$: an $n \times n$ state transition matrix;

$Q(\cdot,\cdot)$: an $n \times m$ matrix;

$H(\cdot,\cdot)$: a $p \times n$ observation matrix;

and

$R(\cdot,\cdot)$: a $p \times p$ nonsingular matrix.

Here, the Markov chain $\gamma(k)$ characterizes, for example, system-component failures, interrupted observation mechanisms, switching environments and so on. Therefore, system (2.1) and (2.2) includes those treated by Nahi (1969), Ackerson and Fu (1970), Dajani and Sage (1970), Jaffer and Gupta (1971), and Sawaragi, Katayama and Fujishige (1972), as special cases. We assume that the transition probabilities of the Markov chain $\gamma(k)$ is known and given by

$$P(\gamma(k)=j|\gamma(k-1)=i) = p_{ij}(k), \quad i, j = 1, 2, \ldots, M \tag{2.3$_1$}$$

and the initial probability distribution is given by

$$P(\gamma(0)=i) = p_i(0), \quad i = 1, 2, \ldots, M. \tag{2.3$_2$}$$

The initial state $x(0)$ is assumed to be Gaussian with mean $\hat{x}(0|-1) = E\{x(0)\}$ and covariance matrix $\hat{P}(0|-1) = Cov\{x(0)\}$, where $E\{\cdot\}$ and $Cov\{\cdot\}$ denote mathematical expectation and covariance matrix of random variables specified in the brackets, respectively. It is further assumed that random variables $w(k)$, $v(k)$, $\gamma(k)$ and $x(0)$ are mutually independent.

The objective of this chapter is to find out the sequential estimator that produces the minimum-variance estimate $\hat{x}(k|k)$ and its approximate estimate $x^*(k|k)$ of the state $x(k)$ by observing the data $\{y(0),y(1),\ldots,y(k)\}$. It is well known (Kalman, 1963; Bucy and Joseph, 1968) that the best estimate which minimizes the Bayes' risk:

$$E\{[x(k) - \hat{x}(k|k)]'V[x(k) - \hat{x}(k|k)]|Y^k\} \tag{2.4}$$

is given by the conditional expectation:

$$\hat{x}(k|k) = E\{x(k)|Y^k\}. \tag{2.5}$$

Here, V is an $n \times n$ symmetric positive definite matrix; Y^k is the minimal σ-field determined by observations $\{y(0), y(1), \ldots, y(k)\}$; $E\{\cdot|Y^k\}$ denotes the conditional expectation given Y^k (Doob, 1953); and the prime denotes the matrix transpose. Hence, if the conditional probability density function $p(x(k)|Y^k)$ of $x(k)$ given Y^k is obtained, the optimal estimate $\hat{x}(k|k)$ can be computed from

$$\hat{x}(k|k) = \int_{R^n} x(k) p(x(k)|Y^k) dx(k) \tag{2.6}$$

where R^n is the n-dimensional Euclidean space.

2.3 Optimal Estimator Algorithm

We shall show first the optimal estimator algorithm for system (2.1) and (2.2), and then the derivation of the optimal algorithm.

Optimal Estimator Algorithm I :

The minimum-variance estimate $\hat{x}(k|k)$ for system (2.1) and (2.2) is given by

$$\hat{x}(k|k) = \sum_{i_0=1}^{M} \cdots \sum_{i_k=1}^{M} \hat{x}(k|k, I^k) P(I^k|Y^k), \tag{3.1}$$

where $I^k = \{i_0, i_1, \ldots, i_k\}$. In (3.1), $\hat{x}(k|k, I^k)$ is given by the following equations:

$$\hat{x}(k|k,I^k) = \hat{x}(k|k-1,I^k) + K(k,I^k)[y(k) - H(k,i_k)\hat{x}(k|k-1,I^k)] \tag{3.2}$$

$$\hat{x}(k|k-1,I^k) = F(k-1,i_k)\hat{x}(k-1|k-1,I^{k-1}) \tag{3.3}$$

where $K(k,I^k)$ in (3.2) is given by

$$K(k,I^k) = \hat{P}(k|k-1,I^k)H'(k,i_k)[H(k,i_k)\hat{P}(k|k-1,I^k)H'(k,i_k) + R(k,i_k)R'(k,i_k)]^{-1} \tag{3.4}$$

$$\hat{P}(k|k,I^k) = \hat{P}(k|k-1,I^k) - K(k,I^k)H(k,i_k)\hat{P}(k|k-1,I^k) \tag{3.5}$$

$$\hat{P}(k|k-1,I^k) = F(k-1,i_k)\hat{P}(k-1|k-1,I^{k-1})F'(k-1,i_k) + Q(k-1,i_k)Q'(k-1,i_k). \tag{3.6}$$

Also, the conditional probability $P(I^k|Y^k)$ appearing in (3.1) is given by

$$P(I^k|Y^k) = \frac{p(y(k)|I^k,Y^{k-1})P(I^k|Y^{k-1})}{\sum_{i_0=1}^{M} \cdots \sum_{i_k=1}^{M} p(y(k)|I^k,Y^{k-1})P(I^k|Y^{k-1})}, \tag{3.7}$$

where $p(y(k)|I^k,Y^{k-1})$ is Gaussian with

$$\text{mean} = H(k,i_k)\hat{x}(k|k-1,I^k) \tag{3.8$_1$}$$

$$\text{cov} = H(k,i_k)\hat{P}(k|k-1,I^k)H'(k,i_k) + R(k,i_k)R'(k,i_k), \quad (3.8)_2$$

and the a priori probability $P(I^k|Y^{k-1})$ appearing in the right-hand side of (3.7) is given by

$$P(I^k|Y^{k-1}) = p_{i_k,i_{k-1}}(k)P(I^{k-1}|Y^{k-1}). \tag{3.9}$$

Remark : We see from (3.1) and (3.7) that the evaluation of the optimal estimate $\hat{x}(k|k)$ requires the evergrowing amount of memory, which makes the practical implementation of the optimal estimator algorithm impossible for large values of time k.

Derivation of the Optimal Estimator Algorithm I :

By the smoothing property of conditional expectations (Doob, 1953), the conditional probability density function $p(x(k)|Y^k)$ in (2.6) can be expressed as

$$\begin{aligned} p(x(k)|Y^k) &= E\{p(x(k)|\gamma(0),\gamma(1),\ldots,\gamma(k),Y^k)|Y^k\} \\ &= \sum_{i_0=1}^{M}\cdots\sum_{i_k=1}^{M} p(x(k)|\gamma(0)=i_0,\ldots,\gamma(k)=i_k,Y^k) \\ &\qquad \times P(\gamma(0)=i_0,\ldots,\gamma(k)=i_k|Y^k). \end{aligned} \tag{3.10}$$

Here, since the values of the Markov chain $\{\gamma(0),\gamma(1),\ldots,\gamma(k)\}$ are specified as conditioning, the conditional probability density function $p(x(k)|\gamma(0)=i_0,\ldots,\gamma(k)=i_k,Y^k)$ is Gaussian with

$$\text{mean} = \hat{x}(k|k,I^k) \triangleq E\{x(k)|\gamma(0)=i_0,\ldots,\gamma(k)=i_k,Y^k\} \quad (3.11)_1$$

$$\text{cov} = \hat{P}(k|k,I^k) \triangleq \text{Cov}\{x(k)|\gamma(0)=i_0,\ldots,\gamma(k)=i_k,Y^k\}, \quad (3.11)_2$$

where $I^k = \{i_0,\ldots,i_k\}$. The conditional mean $\hat{x}(k|k,I^k)$ can be calculated sequentially, for each sequence I^k, by the usual Kalman-filter algorithm (3.2)-(3.6) using the specified values of I^k.

Next, by use of Bayes' rule (Ho and Lee, 1964), the a posteriori probability $P(\gamma(0)=i_0,\ldots,\gamma(k)=i_k|Y^k)$ is given by (3.7)-(3.9), where

$$P(I^k|Y^k) \triangleq P(\gamma(0)=i_0,\ldots,\gamma(k)=i_k|Y^k).$$

Substituting (3.10) into (2.6) and carrying out the integration, we get from (3.11) the minimum-variance estimate $\hat{x}(k|k)$ given by (3.1). This completes the derivation of the optimal estimator algorithm I.

2.4 Another Form of Optimal Estimator Algorithm

The optimal estimator algorithm for system (2.1) and (2.2) has been shown in the previous section. In this section, we shall show the optimal estimator algorithm in another form, which is useful for obtaining the optimal continuous-time estimator algorithm shown in chapter III.

Notice that the Markov chain $\{\gamma(0),\gamma(1),\ldots,\gamma(k)\}$ can be expressed in terms of initial value $\gamma(0)$ and random sequences $\{\tau_1,\tau_2,\ldots,\tau_k\}$ and $\{j_1,j_2,\ldots,j_k\}$, where

$$\tau_k \triangleq \text{the random time that the kth jump of the Markov chain occurs;} \tag{4.1)$_1$}$$

$$j_k \triangleq \text{the random value taken by the Markov chain at the kth jump, that is, } \gamma(\tau_k). \tag{4.1)$_2$}$$

By using these random variables $\gamma(0)$, τ_k and j_k, we can obtain the optimal estimator algorithm in the following form.

Optimal Estimator Algorithm II

The minimum-variance estimate $\hat{x}(k|k)$ for system (2.1) and (2.2) is given by

$$\hat{x}(k|k) = \sum_{i_0=1}^{M} \hat{x}(k|k,\gamma(0)=i_0,\tau_1>k)P(\gamma(0)=i_0,\tau_1>k|Y^k) +$$

$$+ \sum_{n=1}^{k} \sum_{i_0=1}^{M} \cdots \sum_{\substack{i_n=1 \\ i_n \neq i_{n-1}}}^{M} \sum_{t_1=1}^{k-n+1} \sum_{t_2=t_1+1}^{k-n+2} \cdots \sum_{t_n=t_{n-1}+1}^{k}$$

$$\hat{x}(k|k,\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}>k)$$

$$\times P(\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}>k|Y^k) \qquad (4.2)$$

where $t^n = (t_1,t_2,\ldots,t_n)$, $i^n = (i_1,i_2,\ldots,i_n)$, $\tau^n = (\tau_1,\tau_2,\ldots,\tau_n)$ and $j^n = (j_1,j_2,\ldots,j_n)$. Here, the conditional estimates $\hat{x}(k|k,\gamma(0)=i_0,\tau_1>k)$ and $\hat{x}(k|k,\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}>k)$ are defined by

$$\hat{x}(k|k,\gamma(0)=i_0,\tau_1>k) = E\{x(k)|\gamma(0)=i_0,\tau_1>k,Y^k\} \qquad (4.3)_1$$

and

$$\hat{x}(k|k,\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}>k)$$

$$= E\{x(k)|\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}>k,Y^k\} \qquad (4.3)_2$$

and are given by the usual Kalman-filter algorithms by using the values of $\{\gamma(0),\gamma(1),\ldots,\gamma(k)\}$ specified as conditioning. The a posteriori probabilities $P(\gamma(0)=i_0,\tau_1>k|Y^k)$ and $P(\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}>k|Y^k)$ are given by

$$P(\gamma(0)=i_0,\tau_1>k|Y^k) = \frac{p(y(k)|\gamma(0)=i_0,\tau_1>k,Y^{k-1})P(\gamma(0)=i_0,\tau_1>k|Y^{k-1})}{p(y(k)|Y^{k-1})} \qquad (4.4)$$

and

$$P(\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}>k|Y^k)$$

$$= p(y(k)|\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}>k,Y^{k-1})$$

$$\times\ P(\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}>k|Y^{k-1})/p(y(k)|Y^{k-1}) \quad (4.5)$$

where the conditional probability density function $p(y(k)|\gamma(0)=i_0,$ $\tau_1>k,Y^{k-1})$ is Gaussian with

$$\text{mean} = H(k,i_0)\hat{x}(k|k-1,\gamma(0)=i_0,\tau_1>k) \quad (4.6)_1$$

$$\text{cov} = H(k,i_0)\hat{P}(k|k-1,\gamma(0)=i_0,\tau_1>k)H'(k,i_0)$$

$$+ R(k,i_0)R'(k,i_0), \quad (4.6)_2$$

and also $p(y(k)|\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}>k,Y^{k-1})$ is Gaussian with

$$\text{mean} = H(k,i_n)\hat{x}(k|k-1,\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}>k) \quad (4.7)_1$$

$$\text{cov} = H(k,i_n)\hat{P}(k|k-1,\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}>k)H'(k,i_n)$$

$$+ R(k,i_n)R'(k,i_n). \quad (4.7)_2$$

Moreover, the a priori probabilities $P(\gamma(0)=i_0,\tau_1>k|Y^{k-1})$ and $P(\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}>k|Y^{k-1})$ appearing in (4.4) and (4.5) are given by

$$P(\gamma(0)=i_0,\tau_1>k|Y^{k-1}) = p_{i_0,i_0}(k)P(\gamma(0)=i_0,\tau_1>k-1|Y^{k-1}) \quad (4.8)_1$$

and

$$P(\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}>k|Y^{k-1})$$

$$= p_{i_n,i_n}(k)P(\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}>k-1|Y^{k-1}), \text{ if } t_n \leqq k-1,$$

$$= p_{i_{n-1},i_n}(k)P(\gamma(0)=i_0,\tau^{n-1}=t^{n-1},j^{n-1}=i^{n-1}.\tau_n>k-1|Y^{k-1}),$$

$$\text{if } t_n = k. \quad (4.8)_2$$

Furthermore, the conditional probability density function $p(y(k)|Y^{k-1})$ appearing in (4.4) and (4.5) is given by

$$p(y(k)|Y^{k-1}) = \sum_{i_0=1}^{M} p(y(k)|\gamma(0)=i_0,\tau_1>k,Y^{k-1})P(\gamma(0)=i_0,\tau_1>k|Y^{k-1})$$

$$+ \sum_{n=1}^{k} \sum_{i_0=1}^{M} \cdots \sum_{\substack{i_n=1 \\ i_n\neq i_{n-1}}}^{M} \sum_{t_1=1}^{k-n+1} \sum_{t_2=t_1+1}^{k-n+2} \cdots \sum_{t_n=t_{n-1}+1}^{k}$$

$$p(y(k)|\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}>k,Y^{k-1})$$

$$\times P(\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}>k|Y^{k-1}). \quad (4.9)$$

Derivation of Optimal Estimator Algorithm II :

Similarly as in the derivation of the optimal estimator algorithm I, equation (4.2) follows from the smoothing property of conditional expectations (Doob, 1953). The conditional estimates $\hat{x}(k|k,\gamma(0)=i_0,\tau_1>k)$ and $\hat{x}(k|k,\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}>k)$ defined by (4.3) are given by the usual Kalman-filter algorithms; because the values of the Markov chain are specified as conditioning. The a posteriori probabilities $P(\gamma(0)=i_0,\tau_1>k|Y^k)$ and $P(\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}>k|Y^k)$ are obtained by applying Bayes' rule (Ho and Lee, 1964), and are given by equations (4.4)-(4.9).

Remark : Optimal estimator algorithm II is almost the same as optimal algorithm I and is obtained merely by replacing Markov chain $\{\gamma(0),\gamma(1),...\}$ by random sequence $\{\gamma(0),\tau_1,j_1,\tau_2,j_2,...\}$. But the idea for optimal algorithm II is useful for us to obtain the optimal estimator algorithm for continuous systems, which will be made clear in chapter III. It should, however, be noted that optimal estimator algorithms I and II require the evergrowing amount of memory, so that approximate estimator algorithms are to be developed for practical implementation. Finally, it should be noted that, for the case when only a small number of jumps occur, optimal estimator algorithm II is more effective than optimal algorithm I.

2.5 Approximate Estimator Algorithms

In this section, we shall first show an approximate estimator algorithm based upon optimal algorithm I; and later, another approximate estimator will be shown based upon optimal algorithm II.

2.5.1 Approximate Estimator Algorithm I

Optimal algorithms I and II are infeasible because the ever-growing amount of memory is required; so that a feasible approximate estimator algorithm will be derived under the following assumption.

Assumption : The conditional probability density function $p(x(k-1)|Y^{k-1})$ of $x(k-1)$ given Y^{k-1} is Gaussian with

$$E\{x(k-1)|Y^{k-1}\} = x^*(k-1|k-1) \qquad (5.1)_1$$

$$\mathrm{Cov}\{x(k-1)|Y^{k-1}\} = P^*(k-1|k-1) \qquad (5.1)_2$$

for all k.

This kind of assumption has already been used in deriving the approximate nonlinear filters for the usual nonlinear estimation problems (Jazwinski, 1970).

Based upon assumption (5.1), an approximate estimator algorithm can be obtained as follows.

Approximate Estimator Algorithm I

The approximate estimate $x^*(k|k)$ of state $x(k)$ given observations Y^k is furnished by

$$x^*(k|k) = \sum_{i=1}^{M} x^*_i(k|k)p(i|k), \tag{5.2}$$

where

$$x^*_i(k|k) = x^*_i(k|k-1) + K_i(k)[y(k) - H(k,i)x^*_i(k|k-1)] \tag{5.3}$$

$$x^*_i(k|k-1) = F(k-1,i)x^*(k-1|k-1) \tag{5.4}$$

$$K_i(k) = P_i^*(k|k-1)H'(k,i)[H(k,i)P_i^*(k|k-1)H'(k,i) + R(k,i)R'(k,i)]^{-1} \tag{5.5}$$

$$P_i^*(k|k-1) = F(k-1,i)P^*(k-1|k-1)F'(k-1,i) + Q(k-1,i)Q'(k-1,i) \tag{5.6}$$

and

$$p(i|k) = \frac{\Lambda_i(k) \sum_{j=1}^{M} p_{ji}(k)p(j|k-1)}{\sum_{i=1}^{M} \Lambda_i(k) \sum_{j=1}^{M} p_{ji}(k)p(j|k-1)} . \tag{5.7}$$

Here,

$$\Lambda_i(k) = p(y(k)|\gamma(k)=i,Y^{k-1}) \tag{5.8}$$

and the probability density function $p(y(k)|\gamma(k)=i,Y^{k-1})$ is Gaussian with

$$\text{mean} = H(k,i)x^*_i(k|k-1) \qquad (5.9)_1$$

$$\text{cov} = H(k,i)P^*_i(k|k-1)H'(k,i) + R(k,i)R'(k,i). \qquad (5.9)_2$$

Moreover, the approximate error covariance matrix $P^*(k|k)$ is given by

$$P^*(k|k) = \sum_{i=1}^{M}\Big\{[x^*(k|k) - x^*_i(k|k)][x^*(k|k) - x^*_i(k|k)]' + P^*_i(k|k)\Big\}p(i|k), \tag{5.10}$$

where

$$P^*_i(k|k) = P^*_i(k|k-1) - K_i(k)H(k,i)P^*_i(k|k-1). \tag{5.11}$$

Derivation of Approximate Estimator Algorithm I

Rewriting (3.10), we have

$$p(x(k)|Y^k) = \sum_{i=0}^{M} p(x(k)|\gamma(k)=i,Y^k)P(\gamma(k)=i|Y^k). \tag{5.12}$$

This expression is convenient for deriving the approximate sequential estimator algorithm as shown below. Let us first evaluate the conditional probability density function $p(x(k)|\gamma(k)=i,Y^k)$. By use of Bayes' rule (Ho and Lee, 1964), we can see that

$$p(x(k)|\gamma(k)=i,Y^k) = \frac{p(y(k)|\gamma(k)=i,x(k))p(x(k)|\gamma(k)=i,Y^{k-1})}{p(y(k)|\gamma(k)=i,Y^{k-1})} . \tag{5.13}$$

From (2.2), $p(y(k)|\gamma(k)=i,x(k))$ is Gaussian. Therefore, from (5.13), we see under Assumption (5.1) that $p(x(k)|\gamma(k)=i,Y^k)$ is also Gaussian with mean and covariance given by (5.3) and (5.11), respectively, where in (5.3) and (5.11) gain matrix $K_i(k)$ is given by (5.5), and

$$x^*_i(k|k-1) = E\{x(k)|\gamma(k)=i,Y^{k-1}\} \tag{5.14$_1$}$$

$$P^*_i(k|k-1) = Cov\{x(k)|\gamma(k)=i,Y^{k-1}\}. \tag{5.14$_2$}$$

For each i, (5.3) is the same as the usual Kalman-filter algorithm, and $K_i(k)$ defined by (5.5) is the associated optimal gain matrix. From (5.12) and (2.6), the approximate minimum-variance estimate $x^*(k|k)$ is given by (5.2).

Moreover, the associated error covariance matrix:

$$P^*(k|k) = E\{[x(k) - x^*(k|k)][x(k) - x^*(k|k)]'|Y^k\}$$

can be expressed as

$$P^*(k|k) = E\Big\{[x^*(k|k) - x^*_{\gamma(k)}(k|k)][x^*(k|k) - x^*_{\gamma(k)}(k|k)]' + \mathrm{Cov}\{x(k)|\gamma(k),Y^k\}|Y^k\Big\}$$

which becomes (5.10) under Assumption (5.1).

Now consider the conditional probability $P(\gamma(k)=i|Y^k)$. To simplify the notation, we define

$$p(i|k) = P(\gamma(k)=i|Y^k), \quad i = 1, 2,\ldots, M. \tag{5.15}$$

Application of Bayes' rule gives

$$p(i|k) = \frac{p(y(k)|\gamma(k)=i,Y^{k-1})P(\gamma(k)=i|Y^{k-1})}{\sum\limits_{i=1}^{M} p(y(k)|\gamma(k)=i,Y^{k-1})P(\gamma(k)=i|Y^{k-1})}, \tag{5.16}$$

where from Assumption (5.1) the conditional probability density function $p(y(k)|\gamma(k)=i,Y^{k-1})$ is Gaussian with mean and covariance given by (5.9). Also, we see from (2.3) that the a priori probability $P(\gamma(k)=i|Y^{k-1})$ in (5.16) is given by

$$P(\gamma(k)=i|Y^{k-1}) = \sum_{j=1}^{M} p_{ji}(k)P(\gamma(k-1)=j|Y^{k-1}). \tag{5.17}$$

Substituting (5.17) into (5.16) yields (5.7)-(5.9) by definition (5.14).

2.5.2 Approximate Estimator Algorithm II

In this subsection, we shall present an approximate estimator algorithm which corresponds to optimal estimator algorithm II proposed in section 2.4. We assume here that the time interval of interest is finite and the final time is equal to T.

Let us define the number of jumps which occur in the time interval $[0,k]$ as

$$n_\tau^k = \max\{n \mid \tau_n \in \{1,2,\ldots,k\}\}.$$

Then for a small positive number ε, define N_0 as a positive integer such that if $N \geqq N_0$ the inequality

$$P(n_\tau^T \leqq N) > 1 - \varepsilon \tag{5.18}$$

is satisfied. In other words, the Markov chain $\gamma(k)$ jumps more than N_0 times in the time interval $[0,T]$ with probability less than ε. Here, since the discrete system is considered, the integer N_0 defined above is less than T for any ε. Therefore, we can obtain the optimal estimate of $x(k)$ with probability greater than $1 - \varepsilon$, if we take

$$x^*(k|k) = E\{x(k) \mid Y^k, n_\tau^k \leqq N_0\} \tag{5.19}$$

as an estimate of $x(k)$.

By the smoothing property of conditional expectations, equation (5.19) can be expressed as

$$
\begin{aligned}
x^*(k|k) &= E\{E\{x(k)|\gamma(0),\tau^{N_0},j^{N_0},Y^k\}|Y^k,n^k_\tau \le N_0\} \\
&= \sum_{i_0=1}^{M} \hat{x}(k|k,\gamma(0)=i_0,\tau_1>k)P(\gamma(0)=i_0,\tau_1>k|Y^k) \\
&+ \sum_{n=1}^{\min(k,N_0)} \sum_{i_0=1}^{M} \cdots \sum_{\substack{i_n=1 \\ i_n \ne i_{n-1}}}^{M} \sum_{t_1=1}^{k-n+1} \sum_{t_2=t_1+1}^{k-n+2} \cdots \sum_{t_n=t_{n-1}+1}^{k} \\
&\qquad \hat{x}(k|k,\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}>k) \\
&\qquad \times P(\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}>k|Y^k),
\end{aligned}
\tag{5.20}
$$

where $\hat{x}(k|k,\gamma(0)=i_0,\tau_1>k)$ and $\hat{x}(k|k,\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}>k)$ are defined by (4.3), and $\min(a,b)$ denotes the minimum of a and b. The approximate estimate $x^*(k|k)$ is given by (5.20) together with (4.3)-(4.9) where in (4.9) the summation with respect to n should be performed up to $\min(k,N_0)$ terms similarly as in (5.20).

Remark 1 : We have proposed two approximate estimator algorithms for discrete system (2.1) and (2.2). The approximate estimator algorithm I presented in section 2.5.1 has a simple structure as compared with the approximate estimator algorithm II proposed in this subsection. The feasibility of the approximate estimator algorithm I will be demonstrated by numerical examples in section 2.8. If the more accurate estimate of $x(k)$ is desired, the approximate algorithm II may be recommendable and positive number ε in (5.18) should be chosen closely to zero. The idea for the approximate estimator algorithm II is useful for the derivation of an approximate estimator algorithm for continuous systems.

Remark 2 : In deriving the approximate estimator algorithm I, Gaussian assumption (5.1) is used. This kind of assumption is frequently employed in deriving approximate nonlinear filters for usual nonlinear systems (Jazwinski, 1970); however, the justification of the assumption has not yet been made. The only related works are the comparative studies on several nonlinear approximate filters (Schwartz and Stear, 1968; Wishner et al., 1969).

2.6 Adaptive Estimation with Unknown Transition Probability of Markov Chain

In this section, we consider the Bayesian estimation problem for linear discrete systems with a stationary Markov chain, where the values of the transition probabilities of the Markov chain are unknown but fixed. For simplicity, a two-valued Markov chain taking on values of 0 or 1 is considered in the sequel; but the extension to the case of the general multi-valued Markov chain is immediate.

Let us consider the system represented by (2.1) and (2.2), where the Markov chain $\gamma(k)$ is stationary and the transition probabilities do not depend upon time k, that is,

$$P(\gamma(k)=j \mid \gamma(k-1)=i) = p_{ij}, \qquad i, j = 0, 1. \tag*{(6.1)$_1$}$$

Let P be the transition probability matrix of the Markov chain $\gamma(k)$:

$$P = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} . \tag*{(6.1)$_2$}$$

and let the initial distribution be given by

$$\pi = [p_0, p_1]$$

where $p_i = P(\gamma(0)=i)$, $i = 0, 1$. It should be noted that the transition probability matrix P is characterized by the two values p_{00} and p_{11}. It is assumed that p_{00} and p_{11} are unknown constants whose a priori joint probability density function is uniformly distributed over $[0,1] \times [0,1]$; that is,

$$p(p_{00},p_{11}) = 1 , \quad \text{for} \quad (p_{00},p_{11}) \in [0,1] \times [0,1]. \quad (6.2)$$

The problem is to find the adaptive estimator algorithm that sequentially produces the approximate minimum-variance estimate $x^*(k|k)$ of state $x(k)$ by observing the data $\{y(0),y(1), \ldots,y(k)\}$, and that can be applied when the values of the transition probabilities of the Markov chain $\gamma(k)$ are unknown. In order to derive the sequential adaptive estimator algorithm, we employ the same assumption as (5.1).

The adaptive estimator algorithm is then given as follows.

Adaptive Estimator Algorithm

The approximate minimum-variance estimate $x^*(k|k)$ for system (2.1) and (2.2) with unknown transition probability matrix P is given by

$$x^*(k|k) = \sum_{i=0}^{1} x^*_i(k|k)p(i|k), \quad (6.3)$$

where

$$x^*_i(k|k) = x^*_i(k|k-1) + K_i(k)[y(k) - H(k,i)x^*_i(k|k-1)] \quad (6.4)$$

$$x^*_i(k|k-1) = F(k-1,i)x^*(k-1|k-1) \quad (6.5)$$

$$K_i(k) = P^*_i(k|k-1)H'(k,i)[H(k,i)P^*_i(k|k-1)H'(k,i) + R(k,i)R'(k,i)]^{-1} \quad (6.6)$$

$$P_i^*(k|k-1) = F(k-1,i)P^*(k-1|k-1)F'(k-1,i) + Q(k-1,i)Q'(k-1,i) \tag{6.7}$$

and

$$p(1|k) = \frac{\Lambda_1(k) \sum_{i=0}^{1} \overline{p}_{i1}(k-1)p(i|k-1)}{\sum_{j=0}^{1} \Lambda_j(k) \sum_{i=0}^{1} \overline{p}_{ij}(k-1)p(i|k-1)} \tag{6.8}$$

$$p(0|k) = 1 - p(1|k).$$

Here, in (6.8),

$$\Lambda_j(k) \triangleq p(y(k)|\gamma(k)=j,\ Y^{k-1}) \tag{6.9}$$

and the probability density function $p(y(k)|\gamma(k)=j,Y^{k-1})$ is Gaussian with

$$\text{mean} = H(k,j)x^*_j(k|k-1) \qquad (6.10)_1$$

$$\text{cov} = H(k,j)P_j^*(k|k-1)H'(k,j) + R(k,j)R'(k,j). \qquad (6.10)_2$$

Also, $\overline{p}_{ii}(k-1)$ $(i=0,1)$ in (6.8) are given by

$$\overline{p}_{ii}(k-1) = \int_0^1 p_{ii}p(p_{ii}|Y^{k-1})dp_{ii}\ , \quad i = 0,\ 1, \tag{6.11}$$

where

$$p(p_{00}|Y^k) = \frac{\sum_{j=0}^{1} \Lambda_j(k)[p_{0j}p(0|k-1) + \bar{p}_{1j}(k-1)p(1|k-1)]p(p_{00}|Y^{k-1})}{\sum_{j=0}^{1} \Lambda_j(k) \sum_{i=0}^{1} \bar{p}_{ij}(k-1)p(i|k-1)} \tag{6.12}$$

and

$$p(p_{11}|Y^k) = \frac{\sum_{j=0}^{1} \Lambda_j(k)[\bar{p}_{0j}(k-1)p(0|k-1) + p_{1j}p(1|k-1)]p(p_{11}|Y^{k-1})}{\sum_{j=0}^{1} \Lambda_j(k) \sum_{i=0}^{1} \bar{p}_{ij}(k-1)p(i|k-1)} . \tag{6.13}$$

Moreover, the approximate error covariance matrix $P^*(k|k)$ is given by

$$P^*(k|k) = \sum_{i=0}^{1} \left\{ [x^*(k|k) - x^*_i(k|k)][x^*(k|k) - x^*_i(k|k)]' + P_i^*(k|k) \right\} p(i|k) , \tag{6.14}$$

where

$$P_i^*(k|k) = P_i^*(k|k-1) - K_i(k)H(k,i)P_i^*(k|k-1). \tag{6.15}$$

The initial conditions are as follows:

$$x^*(0|-1) = E\{x(0)\}; \quad P^*(0|-1) = \mathrm{Cov}\{x(0)\};$$

$$p(p_{00}|Y^0) = p(p_{11}|Y^0) = 1, \quad 0 \leq p_{00}, p_{11} \leq 1;$$

$$p(1|0) = \frac{\Lambda_1(0)p_1}{\Lambda_1(0)p_1 + \Lambda_0(0)p_0} .$$

Derivation of Adaptive Estimator Algorithm

The minimum-variance estimate $x^*(k|k)$ is given by the conditional expectation:

$$x^*(k|k) = E\{x(k)|Y^k\} = \int_{R^n} x(k)p(x(k)|Y^k)dx(k). \tag{6.16}$$

By the smoothing property of conditional expectations (Doob, 1953),

$$\begin{aligned} p(x(k)|Y^k) &= E\{p(x(k)|\gamma(k),\ldots,\gamma(0),Y^k)|Y^k\} \\ &= \sum_{i_0=0}^{1} \cdots \sum_{i_k=0}^{1} p(x(k)|\gamma(0)=i_0,\ldots,\gamma(k)=i_k,Y^k) \\ &\qquad \times P(\gamma(0)=i_0,\ldots,\gamma(k)=i_k|Y^k). \end{aligned} \tag{6.17}$$

Therefore, the right-hand side of (6.17) is the weighted sum of 2^{k+1} Gaussian probability density functions $p(x(k)|\gamma(0)=i_0,\ldots,\gamma(k)=i_k,Y^k)$. Thus the evaluation of the conditional probability density function $p(x(k)|Y^k)$ through (6.17) is not feasible, because the evergrowing amount of memory is required. To circumvent this difficulty, we derive the approximate estimator algorithm (6.3)-(6.15) under the assumption that the conditional probability density function $p(x(k-1)|Y^{k-1})$ of $x(k-1)$ given Y^{k-1} is Gaussian for all k with

$$\text{mean} = x^*(k-1|k-1) \tag*{(6.18)$_1$}$$

$$\text{cov} = P^*(k-1|k-1). \tag*{(6.18)$_2$}$$

Rewriting (6.17), we get

$$p(x(k)|Y^k) = \sum_{i=0}^{1} p(x(k)|\gamma(k)=i,Y^k)P(\gamma(k)=i|Y^k) \ . \tag{6.19}$$

Let us first consider the conditional probability density function $p(x(k)|\gamma(k)=i,Y^k)$. By use of Bayes' rule, we can see under assumption (6.18) that $p(x(k)|\gamma(k)=i,Y^k)$ is Gaussian with mean $x^*_i(k|k)$ and covariance matrix $P^*_i(k|k)$, where $x^*_i(k|k)$ and $P^*_i(k|k)$ are given by (6.4)-(6.7) and (6.15).

Next, let us consider the a posteriori probability $P(\gamma(k)=i|Y^k)$ appearing in (6.19). Define

$$p(i|k) = P(\gamma(k)=i|Y^k) \ , \quad i = 0,\ 1. \tag{6.20}$$

By the smoothing property of conditional expectations,

$$\begin{aligned} p(1|k) &= E\{P(\gamma(k)=1|p_{00},p_{11},Y^k)|Y^k\} \\ &= \int_0^1\int_0^1 P(\gamma(k)=1|p_{00},p_{11},Y^k)p(p_{00},p_{11}|Y^k)dp_{00}dp_{11} \end{aligned} \tag{6.21}$$

and

$$p(0|k) = 1 - p(1|k).$$

In order to obtain a feasible sequential estimator algorithm, it is assumed that the information for $x(k-1)$ and $\gamma(k-1)$ given Y^{k-1}, that is, (6.18) and $p(1|k-1)$ is complete, and the one-stage filtering will be repeated at each sample time. Thus it follows from Bayes' rule and assumption (6.18) that

$$P(\gamma(k)=1|p_{00},p_{11},Y^k) = \frac{\Lambda_1(k)\sum_{i=0}^{1} p_{i1}p(i|k-1)}{\sum_{j=0}^{1}\Lambda_j(k)\sum_{i=0}^{1} p_{ij}p(i|k-1)} \tag{6.22}$$

and

$$p(p_{00},p_{11}|Y^k) = \frac{\sum_{j=0}^{1}\Lambda_j(k)\sum_{i=0}^{1} p_{ij}p(i|k-1)p(p_{00},p_{11}|Y^{k-1})}{\sum_{j=0}^{1}\Lambda_j(k)\sum_{i=0}^{1} \overline{p}_{ij}(k-1)p(i|k-1)}, \tag{6.23}$$

where $\Lambda_j(k) \triangleq p(y(k)|\gamma(k)=j,Y^{k-1})$ is Gaussian with mean and covariance given by (6.10), and

$$\overline{p}_{ij}(k-1) \triangleq E\{p_{ij}|Y^{k-1}\} = \int_0^1\int_0^1 p_{ij}p(p_{00},p_{11}|Y^{k-1})dp_{00}dp_{11}, \tag{6.24}$$

that is, $\overline{p}_{ij}(k-1)$ $(i,j=0,1)$ are the estimates of the unknown transition probabilities p_{ij} based upon Y^{k-1}. Substituting (6.22) and (6.23) into (6.21) and performing the integration yields equation (6.8). It should be noted that for the evaluation of the conditional probability $p(1|k)$ the values of the conditional means $\overline{p}_{ii}(k-1)$ $(i=0,1)$ are required in (6.8). By assuming that

$$p(p_{00},p_{11}|Y^{k-1}) = p(p_{00}|Y^{k-1})p(p_{11}|Y^{k-1})$$

for the joint conditional probability density function $p(p_{00}, p_{11}|Y^{k-1})$, the integration of (6.23) with respect to p_{11} and p_{00}, respectively, yields (6.12) and (6.13).

Moreover, since the conditional covariance matrix

$$P^*(k|k) = E\{[x(k) - x^*(k|k)][x(k) - x^*(k|k)]'|Y^k\}$$

is expressed as

$$P^*(k|k) = \sum_{i=0}^{1} p(i|k)E\{[x(k) - x^*(k|k)] \\ \times [x(k) - x^*(k|k)]'|\gamma(k)=i,Y^k\} ,$$

equation (6.14) is immediate.

Finally, equation (6.3) can be obtained from (6.19) and (6.16). This completes the derivation of the adaptive estimator algorithm

Remark 1 : Equations (6.12) and (6.13) are, respectively, the recursive relations for the marginal conditional probability density functions and they are related to each other only through the conditional means $\bar{p}_{ii}(k-1)$ $(i=0,1)$. Thus only the storage of the respective marginal conditional probability density functions is required. The conditional means are obtained by numerical integration over [0,1] as

$$\bar{p}_{ii}(k-1) = \int_0^1 p_{ii} p(p_{ii}|Y^{k-1}) dp_{ii} \cong \frac{1}{N} \sum_{s=1}^{N} p_{ii}^s p(p_{ii}^s|Y^{k-1})$$

where $i = 0, 1$, and N denotes the number of grid points in $[0,1]$.

Remark 2 : It should be noted that, if the values of the transition probabilities p_{ij} $(i,j=0,1)$ are known a priori, the non-adaptive sequential algorithm is given by replacing $\bar{p}_{ij}(k-1)$ in (6.8) by the known values of p_{ij} (cf. section 2.5.1).

2.7 Special Case : State Estimation for Linear Discrete System with Interrupted Observation

We have considered, with great generality, the estimation problems for linear discrete systems with a Markov chain. In this section, our attention is focused on linear discrete systems including the Markov chain in a special way, that is, linear discrete systems with interrupted observations.

Let the system be represented by the following linear stochastic difference equations:

$$x(k+1) = F(k)x(k) + Q(k)w(k) \tag{7.1}$$

$$y(k) = \gamma(k)H(k)x(k) + R(k)v(k). \tag{7.2}$$

The difference between the present system (7.1) and (7.2) and the original system (2.1) and (2.2) is that in (7.1) and (7.2) matrices $F(k)$, $Q(k)$ and $R(k)$ do not depend upon the Markov chain $\gamma(k)$; that the observation matrix appearing in (2.2) has the special form $\gamma(k)H(k)$ in (7.2); and that the Markov chain $\gamma(k)$ takes on values of 0 or 1 in the present system. The Markov chain $\gamma(k)$ characterizes the interrupted observation mechanisms and is called interruption process in the sequel.
If the interruption process $\gamma(k)$ is equal to 1, the observation $y(k)$ contains the information about the state $x(k)$; while if $\gamma(k)$ is equal to 0, the observation $y(k)$ contains the noise only.

For system (7.1) and (7.2), the approximate estimator algorithm I presented in section 2.5.1 becomes as follows.

Approximate Estimator Algorithm

The approximate estimate $x^*(k|k)$ of state $x(k)$ based upon observation Y^k is given by

$$x^*(k|k) = x^*(k|k-1) + p(1|k)K(k)[y(k) - H(k)x^*(k|k-1)] \tag{7.3}$$

where

$$x^*(k|k-1) = F(k-1)x^*(k-1|k-1) \tag{7.4}$$

$$K(k) = P^*(k|k-1)H'(k)[H(k)P^*(k|k-1)H'(k) + R(k)R'(k)]^{-1} \tag{7.5}$$

$$P^*(k|k-1) = F(k-1)P^*(k-1|k-1)F'(k-1) + Q(k-1)Q'(k-1) \tag{7.6}$$

and

$$p(1|k) = \frac{\Lambda_1(k) \sum_{i=0}^{1} p_{i1}(k)p(i|k-1)}{\sum_{j=0}^{1} \Lambda_j(k) \sum_{i=0}^{1} p_{ij}(k)p(i|k-1)} \tag{7.7}$$

$$p(0|k) = 1 - p(1|k).$$

Here,

$$\Lambda_j(k) \triangleq p(y(k)|\gamma(k)=j,Y^{k-1}) \ , \quad j = 0,\ 1 \tag{7.8}$$

and the probability density function $p(y(k)|\gamma(k)=j,Y^{k-1})$ is Gaussian with

$$\text{mean} = j\cdot H(k)x^*(k|k-1) \tag*{(7.9)$_1$}$$

$$\text{cov} = j\cdot H(k)P^*(k|k-1)H'(k) + R(k)R'(k). \tag*{(7.9)$_2$}$$

moreover, the approximate error covariance matrix $P^*(k|k)$ is given by

$$P^*(k|k) = P^*(k|k-1) - p(1|k)K(k)H(k)P^*(k|k-1)$$
$$+ p(1|k)[1 - p(1|k)]K(k)T(k)K'(k), \tag{7.10}$$

where

$$T(k) = [y(k) - H(k)x^*(k|k-1)][y(k) - H(k)x^*(k|k-1)]' \tag{7.11}$$

Remark : The simplification of the present algorithm is due to the fact that system equation (7.1) does not depend upon interruption process $\gamma(k)$ and that interruption process $\gamma(k)$ takes on values of 0 or 1 only.

Now, we shall make a further specialization as follows.

Case 1 : The interruption process $\gamma(k)$ is an independent sequence with

$$P(\gamma(k)=1) = p(k) \qquad (7.12)_1$$

$$P(\gamma(k)=0) = 1 - p(k). \qquad (7.12)_2$$

Here, $p(k)$ is the probability that the kth observation conveys the information concerning the state $x(k)$. Consequently, at any time the observation is interrupted with probability $1 - p(k)$.

Case 2 : The interruption process $\gamma(k)$ is constant for all k and is equal to γ, where the value of γ is unknown and

$$P(\gamma=1) = \lambda \qquad (7.13)_1$$

$$P(\gamma=0) = 1 - \lambda. \qquad (7.13)_2$$

Here, the transition probability matrix becomes

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the initial probability distribution is

$$\pi = [1-\lambda, \lambda].$$

For Case 1, the approximate estimator algorithm is given as follows.

Approximate Estimator Algorithm for Case 1

The approximate estimate $x^*(k|k)$ is given by (7.3)-(7.11), where for Case 1 equation (7.7) becomes

$$p(1|k) = \frac{\Lambda_1(k)p(k)}{\Lambda_1(k)p(k) + \Lambda_0(k)[1 - p(k)]} . \tag{7.14}$$

For Case 2, the optimal estimator algorithm presented in sections 2.3 and 2.4 becomes feasible, because the interruption process $\gamma(k)$ does not jump. In fact, equation (3.10) becomes

$$p(x(k)|Y^k) = \sum_{i=0}^{1} p(x(k)|\gamma=i, Y^k)P(\gamma=i\ Y^k), \tag{7.15}$$

and the a posteriori probability density function $p(x(k)|Y^k)$ is the weighted sum of two Gaussian probability density functions for all k. The optimal estimator algorithm in section 2.4 becomes, for Case 2, as follows.

Optimal Estimator Algorithm for Case 2

The minimum-variance estimate $\hat{x}(k|k)$ is given by

$$\hat{x}(k|k) = p(1|k)\hat{x}_1(k|k) + [1 - p(1|k)]\hat{x}_0(k|k), \tag{7.16}$$

where

$$\hat{x}_i(k|k) = \hat{x}_i(k|k-1) + i \cdot K(k)[y(k) - H(k)\hat{x}_1(k|k-1)] \tag{7.17}$$

$$K(k) = \hat{P}_1(k|k-1)H'(k)[H(k)\hat{P}_1(k|k-1)H'(k) + R(k)R'(k)]^{-1} \tag{7.18}$$

$$\hat{P}_i(k|k) = \hat{P}_i(k|k-1) - i \cdot K(k)H(k)\hat{P}_1(k|k-1) \tag{7.19}$$

$$\hat{x}_i(k+1|k) = F(k)\hat{x}_i(k|k) \tag{7.20}$$

$$\hat{P}_i(k+1|k) = F(k)\hat{P}_i(k|k)F'(k) + Q(k)Q'(k), \quad i = 0, 1, \tag{7.21}$$

and

$$p(1|k) = \frac{\Lambda_1(k)p(1|k-1)}{\Lambda_1(k)p(1|k-1) + \Lambda_0(k)[1 - p(1|k-1)]}. \tag{7.22}$$

Here,

$$\Lambda_i(k) \triangleq p(y(k)|\gamma=i, Y^{k-1}) \tag{7.23}$$

and the conditional probability density function $p(y(k)|\gamma=i, Y^{k-1})$ is Gaussian with

$$\text{mean} = i \cdot H(k)\hat{x}_1(k|k-1) \tag{7.24$_1$}$$

$$\text{cov} = i \cdot H(k)\hat{P}_1(k|k-1)H'(k) + R(k)R'(k). \tag{7.24$_2$}$$

Initial conditions are

$$\hat{x}_i(0|-1) = E\{x(0)\}, \quad \hat{P}_i(0|-1) = \text{Cov}\{x(0)\}, \quad i = 0, 1$$

and

$$p(1|-1) = \lambda.$$

Moreover, the estimation error covariance matrix is given by

$$\begin{aligned}\hat{P}(k|k) &\triangleq \text{Cov}\{x(k)|Y^k\} \\ &= p(1|k)\hat{P}_1(k|k) + [1 - p(1|k)]\hat{P}_0(k|k) \\ &\quad + p(1|k)[1 - p(1|k)][\hat{x}_1(k|k) - \hat{x}_0(k|k)] \\ &\qquad \times [\hat{x}_1(k|k) - \hat{x}_0(k|k)]'\end{aligned} \tag{7.25}$$

<u>Remark 1</u> : Nahi (1969) considered the class of linear estimators for the above-mentioned two cases. His result is entirely different from the above result in that the estimator dynamics given by (7.3)-(7.11) and (7.14) for Case 1 and (7.16)-(7.25) for Case 2 is nonlinear with respect to observations and that the corresponding covariance equation is directly related to the observations. It may be noted that for Case 1 if the conditional probability

$p(1|k)$ defined by (5.14) is reduced to 1, then the resultant estimator becomes the usual Kalman filter (Kalman, 1963; Bucy and Joseph, 1968). This shows the plausibility of the proposed estimator algorithm.

Remark 2 : The result for Case 2 substantially agrees with the one by Magil (1965), and the extension of the result to the case of multi-valued γ is immediate. Moreover, regarding the parameter γ as the unknown system parameter, the present result is directly applicable to parameter identification problems (Lainiotis, 1971).

Remark 3 : For the interrupted observation process considered here, we could implement the likelihood ratio test (Van Trees, 1968) in order to detect whether $\gamma(k)$ is 0 or 1; after the decision has been made, we believe that it is absolutely correct. If the answer is $\gamma(k) = 1$, then the observed data will be used by the estimator for the purpose of prediction or filtering. However, associated with the decision made, there is a nonzero probability that the observation contains no information concerning the state to be estimated, that is, a nonzero probability of false alarm. It is clear that this information, which is contained in the conditional probability $p(1|k)$, is useful for improving the performance of the estimator. As is seen from (7.14),

$p(1|k)$ is nonlinear with respect to observations. Therefore, if we restrict ourselves to the class of linear estimators, we can not utilize this information. In view of this, the linear estimators developed by Nahi (1969) may not be very effective.

Furthermore, we shall consider the adaptive estimation problem for the following case.

Case 3 : The situation is the same as in Case 1, except that $p(k)$ defined by (7.12) is constant, that is,

$$p(k) = q, \quad k = 0, 1, 2, \ldots \tag{7.26}$$

and that the value of q is unknown. The a priori probability density function of q is assumed to be given by

$$p(q) = 1, \quad \text{for} \quad 0 \leqq q \leqq 1. \tag{7.27}$$

It is to be noted that from (7.26) there holds

$$p_{11} = 1 - p_{00} = q \tag{7.28}$$

for p_{11} and p_{00} in $(6.1)_2$ and thus that the conditional joint probability density function of p_{00} and p_{11} becomes

$$p(p_{00}, p_{11}|Y^k) = p(p_{11}|Y^k)\delta(p_{00} - (1-p_{11})), \tag{7.29}$$

where $\delta(\cdot)$ is Dirac delta function. Substituting (7.29) into

(6.23) and performing the integration with respect to p_{00} yields

$$p(q|Y^k) = \frac{[\Lambda_1(k)q + \Lambda_0(k)(1 - q)]p(q|Y^{k-1})}{\Lambda_1(k)\bar{q}(k-1) + \Lambda_0(k)[1 - \bar{q}(k-1)]} , \tag{7.30}$$

where q is used in place of p_{00},

$$\bar{q}(k-1) \triangleq E\{q|Y^{k-1}\} = \int_0^1 qp(q|Y^{k-1}), \tag{7.31}$$

and $\Lambda_j(k) = p(y(k)|\gamma(k)=j, Y^{k-1})$ is Gaussian with mean and covariance given by (7.9). Moreover, by the similar procedure as taken in the derivation of (6.8), we have

$$p(1|k) = \frac{\Lambda_1(k)\bar{q}(k-1)}{\Lambda_1(k)\bar{q}(k-1) + \Lambda_0(k)[1 - \bar{q}(k-1)]} . \tag{7.32}$$

Thus the adaptive estimator algorithm for Case 3 becomes as follows.

Adaptive Estimator Algorithm for Case 3

The adaptive sequential estimate $x^*(k|k)$ for Case 3 is given by (7.3)-(7.6) and (7.8)-(7.11) together with (7.30)-(7.32).

Remark 4 : It should be noted that $p(1|k)$ defined by (7.32) is also obtained by replacing $p(k)$ in (7.14) by $\bar{q}(k-1)$ given by (7.31). Therefore, comparing the adaptive estimator algorithm with the nonadaptive one, the increase of the amount of computation in the adaptive case is only the calculation of $\overline{q}(k-1)$.

Remark 5 : It should also be noted that since the approximate estimator algorithms are entirely based upon the assumption that the a priori conditional probability density function $p(x(k)|Y^{k-1})$ is Gaussian, there is a possibility of divergence of the estimation error (Jazwinski, 1970), and the quality of the estimate is not known from the covariance equations defined by (7.6) and (7.10). Therefore, the justification of the assumption must be made in connection with the stability of the proposed estimator algorithms. These are common problems associated with the nonlinear approximate filtering (Jazwinski, 1970). But the evaluation of the accuracy of the approximate nonlinear estimators is in general extremely difficult; there are only a few papers on this problem so far (Schwartz and Stear, 1968; Wishner et al., 1969). Finally, it may be noted that the advantage of Nahi's linear estimators is that it is free from the above-mentioned difficult problems associated with approximate estimators; that is, the quality of the estimate is known a priori from the covariance equations.

We have shown sequential estimator algorithms for several special cases. In the next section, simulation studies will be carried out to demonstrate the feasibility of the proposed estimator algorithms.

2.8 Numerical Example

First, we shall show examples illustrating the application of the proposed estimator algorithms and compare the performance of the proposed nonlinear estimator with that of the best linear estimator due to Nahi (1969) for Case 1 where the interruption process $\gamma(k)$ is an independent sequence.

Consider the scalar system:

$$x(k+1) = ax(k) + Qw(k) \tag{8.1}$$

$$y(k) = \gamma(k)x(k) + Rv(k), \tag{8.2}$$

where $|a| < 1$, and let $p(k) = q$. Here, $p(k)$ is defined by

$$p(k) \triangleq P(\gamma(k)=1),$$

which is the probability that the kth observation conveys the information concerning the state $x(k)$. From (7.5),

$$K(k) = \frac{P^*(k|k-1)}{P^*(k|k-1) + R^2} . \tag{8.3}$$

By using (7.3) and (7.10), we have

$$x^*(k|k) = x^*(k|k-1) + p(1|k)K(k)[y(k) - x^*(k|k-1)], \tag{8.4}$$

$$\begin{aligned} P^*(k|k) &= P^*(k|k-1) - p(1|k)P^*(k|k-1)^2/\Sigma(k) \\ &\quad + p(1|k)[1 - p(1|k)]P^*(k|k-1)^2 \\ &\qquad \times [y(k) - x^*(k|k-1)]^2/\Sigma(k)^2, \end{aligned} \tag{8.5}$$

where

$$\Sigma(k) = P^*(k|k-1) + R^2. \tag{8.6}$$

Also, from (7.4) and (7.6),

$$x^*(k|k-1) = ax^*(k-1|k-1), \tag{8.7}$$

$$P^*(k|k-1) = a^2P^*(k-1|k-1) + Q^2. \tag{8.8}$$

Moreover, from (7.7)-(7.9),

$$p(1|k) = \frac{qR\exp\{-[y(k)-x^*(k|k-1)]^2/2\Sigma(k)\}}{qR\exp\{-[y(k)-x^*(k|k-1)]^2/2\Sigma(k)\} + (1-q)\Sigma^{1/2}(k)\exp\{-y(k)^2(k)/2R^2\}}. \tag{8.9}$$

Digital simulation studies are carried out by using the following set of numerical values:

$$a = 0.95, \quad Q^2 = 0.64, \quad R^2 = 1.69, \quad q = 0.8$$

$$x^*(0|-1) = 10, \quad P^*(0|-1) = 2.$$

The initial value of the state $x(0)$ is sampled randomly at each experiment from the population with $N(30,2)$, whereas the initial estimate $x^*(0|-1)$ is fixed in the whole experiment as above.

Figure 2.1 displays the running values of the estimate $x^*(k|k-1)$, together with the estimate $\bar{x}(k|k-1)$ due to the linear estimator (Nahi, 1969), and the actual state $x(k)$. The covariances are displayed in Fig. 2.2. We observe that if $\gamma(k) = 0$, the covariance $P^*(k|k-1)$ of the next stage takes a large value; on the other hand, since $\bar{P}(k|k-1)$ due to Nahi (1969) is deterministic, it is independent of the actual observations. This is a typical feature of the nonlinear estimator which involves the estimation algorithm (8.9). The performance of the two estimators is compared on the basis of the sample root mean squares:

$$C^*(k) = \left\{\frac{1}{N}\sum_{j=1}^{N}[x^j(k) - x^{*j}(k|k-1)]^2\right\}^{\frac{1}{2}}, \tag{8.10}$$

$$\bar{C}(k) = \left\{\frac{1}{N}\sum_{j=1}^{1}[x^j(k) - \bar{x}^j(k|k-1)]^2\right\}^{\frac{1}{2}}, \tag{8.11}$$

where $C^*(k)$ is the performance index of the nonlinear estimator, and $\overline{C}(k)$ is that of the best linear estimator (Nahi, 1969), and the superscript j denotes the number of simulation run. Total of 50 runs are made in each experiment (N=50); each run has a different noise sample. The results are shown in Figs. 2.3 and 2.4. The difference of the two experiments is that in Fig. 2.4 the sequence $\{\gamma(k), k = 0, 1, \ldots\}$ is fixed through out the experiment. We can observe that the nonlinear estimator indicates significant improvement over the linear estimator, especially in the earlier stages of estimation. This fact is explained as follows. Since, in the earlier stages, the S/N ratio $x^2(k)/R^2$ is extremely high, then whether $\gamma(k) = 0$ or 1 can be detected by (8.9) with high accuracy. Therefore, the present nonlinear estimator may produce nearly optimal estimate.

We now turn to the adaptive case when the actual value of q is unknown, which was treated as Case 3 in the previous section. For the computation of $p(q|Y^k)$ by (7.30), it is approximated by the values on 100 grid points equally spaced in [0,1]. Thus from (7.30),

$$p(q_s|Y^k) = \frac{[q_s\Lambda_1(k) + (1-q_s)\Lambda_0(k)]p(q_s|Y^{k-1})}{\overline{q}(k-1)\Lambda_1(k) + [1 - \overline{q}(k-1)]\Lambda_0(k)} , \qquad (8.12)$$

where $q_s = s/100$, $s = 1, 2, \ldots, 100$, and from (7.31),

$$\overline{q}(k-1) \cong \sum_{s=1}^{100} q_s p(q_s | Y^{k-1})/100 \ . \tag{8.13}$$

Then from (7.32),

$$p(1|k) = \frac{\overline{q}(k-1)\Lambda_1(k)}{\overline{q}(k-1)\Lambda_1(k) + [1 - \overline{q}(k-1)]\Lambda_0(k)} \ . \tag{8.14}$$

Digital simulation studies are also carried out; the sample behavior for $x^*(k|k-1)$ and $P^*(k|k-1)$ is almost the same as those of Figs. 2.1 and 2.2. Figure 2.5 displays the tracking behavior of $\overline{q}(k-1)$ for the unknown value of q, together with the sample average $\overline{\overline{q}}(k-1)$ of $\overline{q}(k-1)$ for 50 simulation runs. We may observe that although the present algorithm is an approximate one, $\overline{q}(k-1)$ serves as an estimate of the value of unknown probability q.

Next, let us consider the adaptive estimation problem for the scalar system described by (8.1) and (8.2) with a Markov chain $\gamma(k)$ whose transition probabilities are unknown, that is, the system treated in section 2.6.

Digital simulation studies are carried out by using the following set of numerical values:

$$a = 0.95, \quad Q^2 = 10, \quad x^*(0|-1) = 30, \quad P^*(0|-1) = 0,$$

$$P = \begin{pmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{pmatrix} \quad \text{(unknown)}, \qquad \pi = [\ 0.1,\ 0.9]\ .$$

The initial value of the state $x(0)$ is assumed to be known at each experiment, and the conditional probability density functions $p(p_{00}|Y^k)$ and $p(p_{11}|Y^k)$ are approximated by the values on 100 grid points equally spaced over [0,1].

The performance of the adaptive and the nonadaptive estimators is compared on the basis of the following indices:

$$C^* = \frac{1}{10} \sum_{j=1}^{10} \left\{ \sum_{k=0}^{800} |x^j(k) - x^{*j}(k|k)| \right\}, \tag{8.15}$$

$$\overline{C} = \frac{1}{10} \sum_{j=1}^{10} \left\{ \sum_{k=0}^{800} |x^j(k) - \overline{x}(k|k)| \right\}, \tag{8.16}$$

where C^* is the performance index of the adaptive estimator presented in section 2.6, $\overline{C}$ is that of the nonadaptive estimator with transition probability matrix:

$$P = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix},$$

and the superscript j denotes the number of simulation run. The result is shown in Fig. 2.6. We can observe that the adaptive estimator indicates the improvement over the nonadaptive estimator; the difference between the performance indices of the two estimators becomes larger as the variance of the observation noise increases. Figure 2.7 displays the tracking behavior of the estimated transition probabilities $\bar{p}_{ii}(k-1)$ for the unknown values of p_{ii} $(i=0,1)$ together with the sample averages $\bar{\bar{p}}_{ii}(k-1)$ of $\bar{p}_{ii}(k-1)$ for 10 simulation runs. We can observe that although the present algorithm is an approximate one, $\bar{p}_{ii}(k-1)$ serve as estimates of the values of the unknown transition probabilities p_{ii} $(i=0,1)$.

2.9 Concluding Remarks

In this chapter, we have derived the optimal estimator algorithm for linear discrete systems with a Markov chain in two forms by characterizing the Markov chain $\{\gamma(k), k=0,1,2,\ldots\}$ in terms of: (1) the instantaneous values of $\gamma(k)$, and (2) the initial value $\gamma(0)$, the jump times τ_k and the values j_k taken at the jump times. The optimal estimator algorithm, however, requires the evergrowing amount of memory, so that the optimal algorithm becomes practically infeasible as time elapses. Therefore, feasible approximate estimator algorithms have been proposed for the practical implementation. Also presented is the adaptive sequential estimator algorithm for the case where the transition probabilities of the Markov chain are unknown but fixed; therefore, the adaptive estimator algorithm presented is coupled with the estimation of the unknown transition probabilities. Digital simulation studies was carried out for a scalar system with interrupted observation mechanisms and demonstrated the feasibility and effectiveness of the proposed estimator algorithms, although the justification of Gaussian assumption (5.1) has not yet been available.

Finally, it should be noted that, even if the actual system contains several Markov chains, we can express these Markov chains in terms of a single Markov chain, so that we can assume without loss of generality that the system contains only a single Markov chain as treated in this chapter.

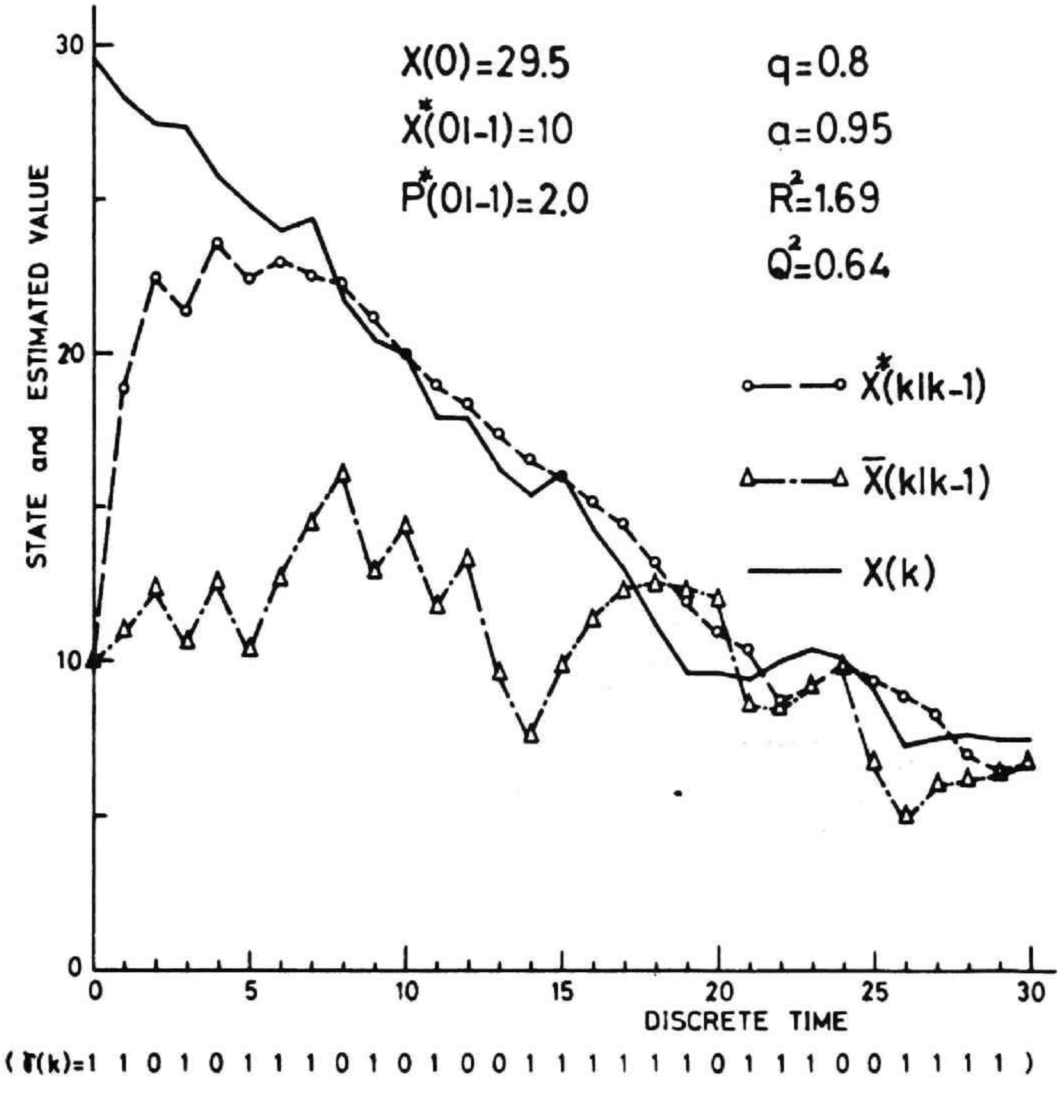

Fig. 2.1. Sample paths of $x^*(k|k-1)$, $\overline{x}(k|k-1)$ and $x(k)$.

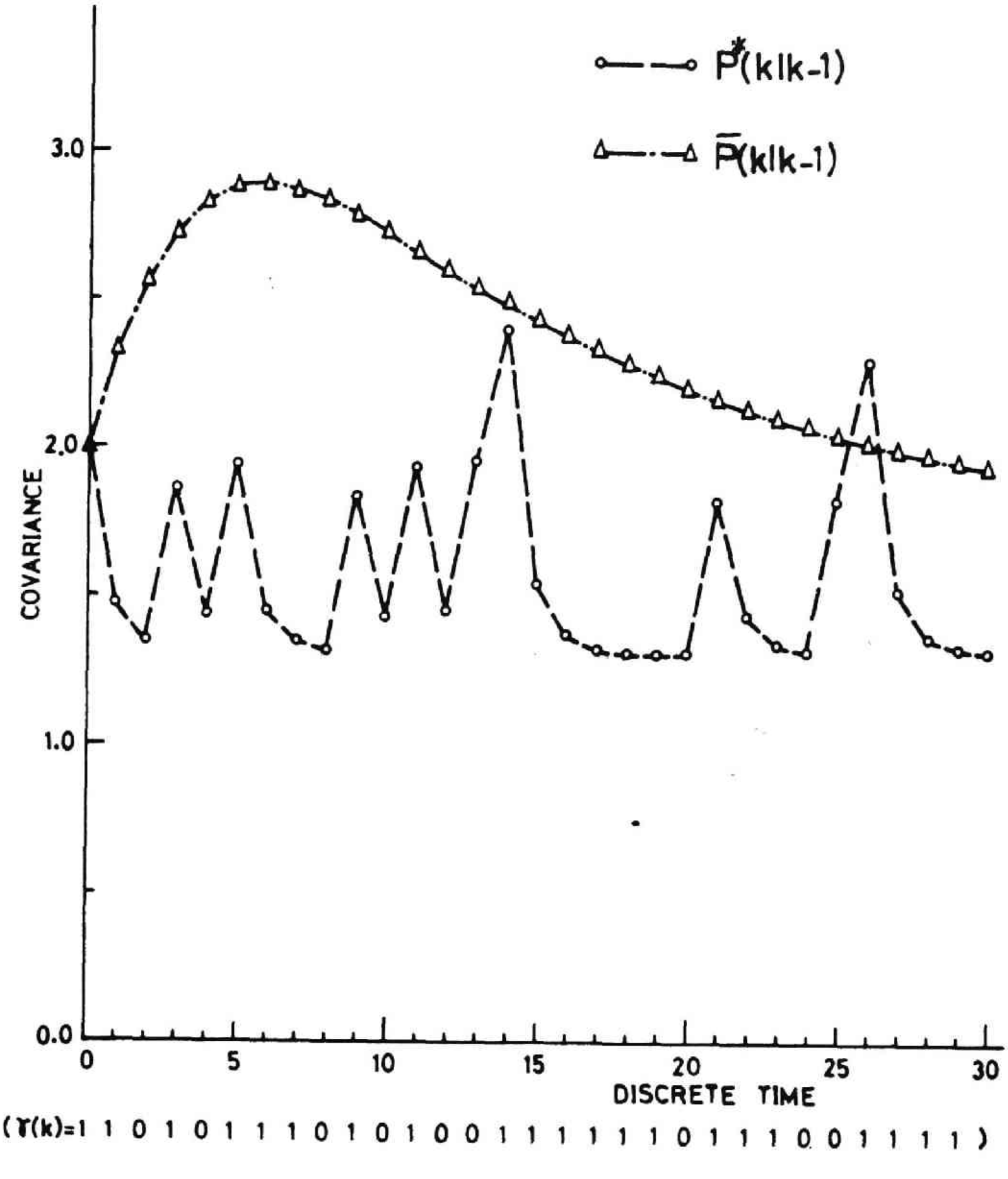

Fig. 2.2. Sample covariances $P^*(k|k-1)$ and $\overline{P}(k|k-1)$.

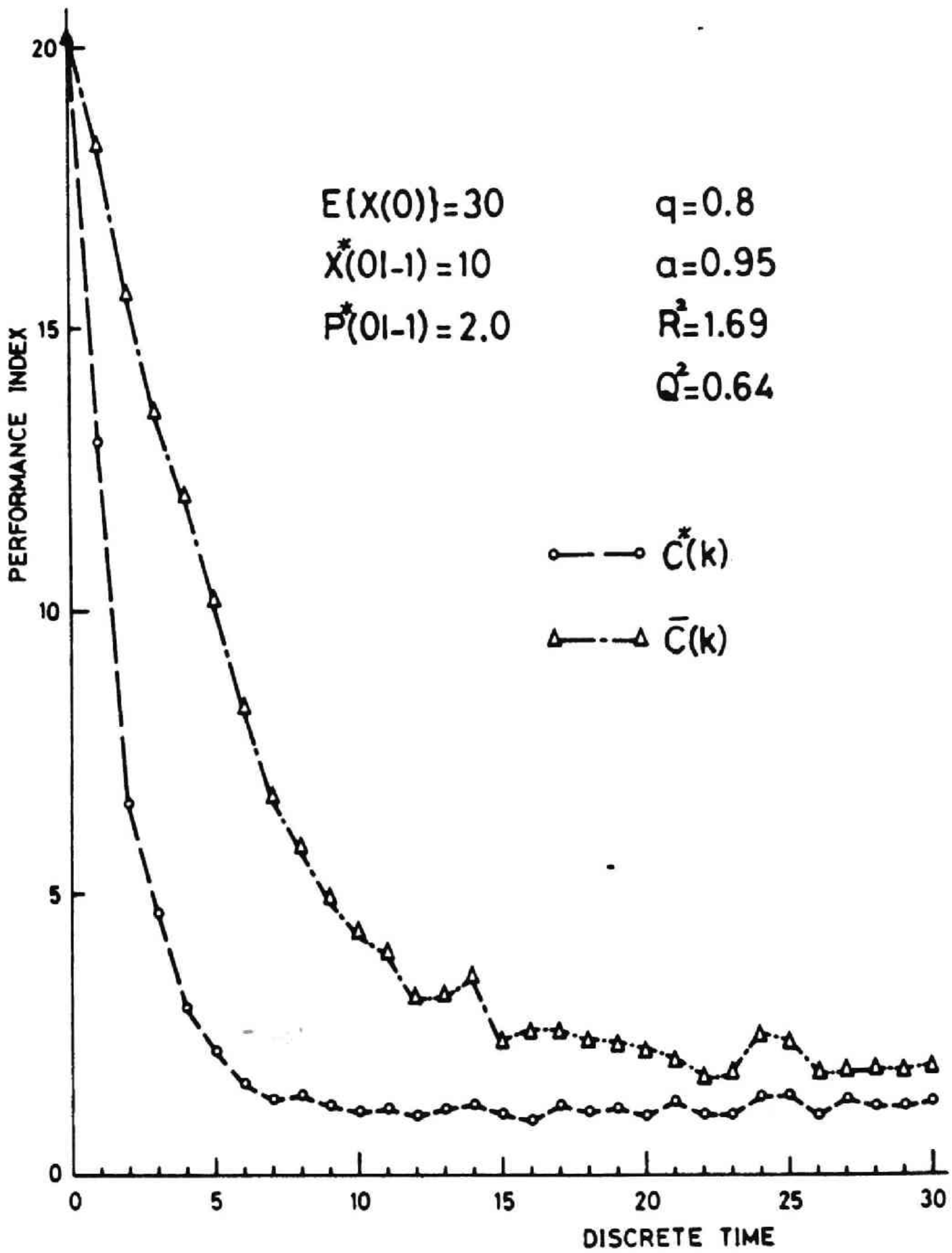

Fig. 2.3. Performance indices $C^*(k)$ and $\overline{C}(k)$ for 50 run average.

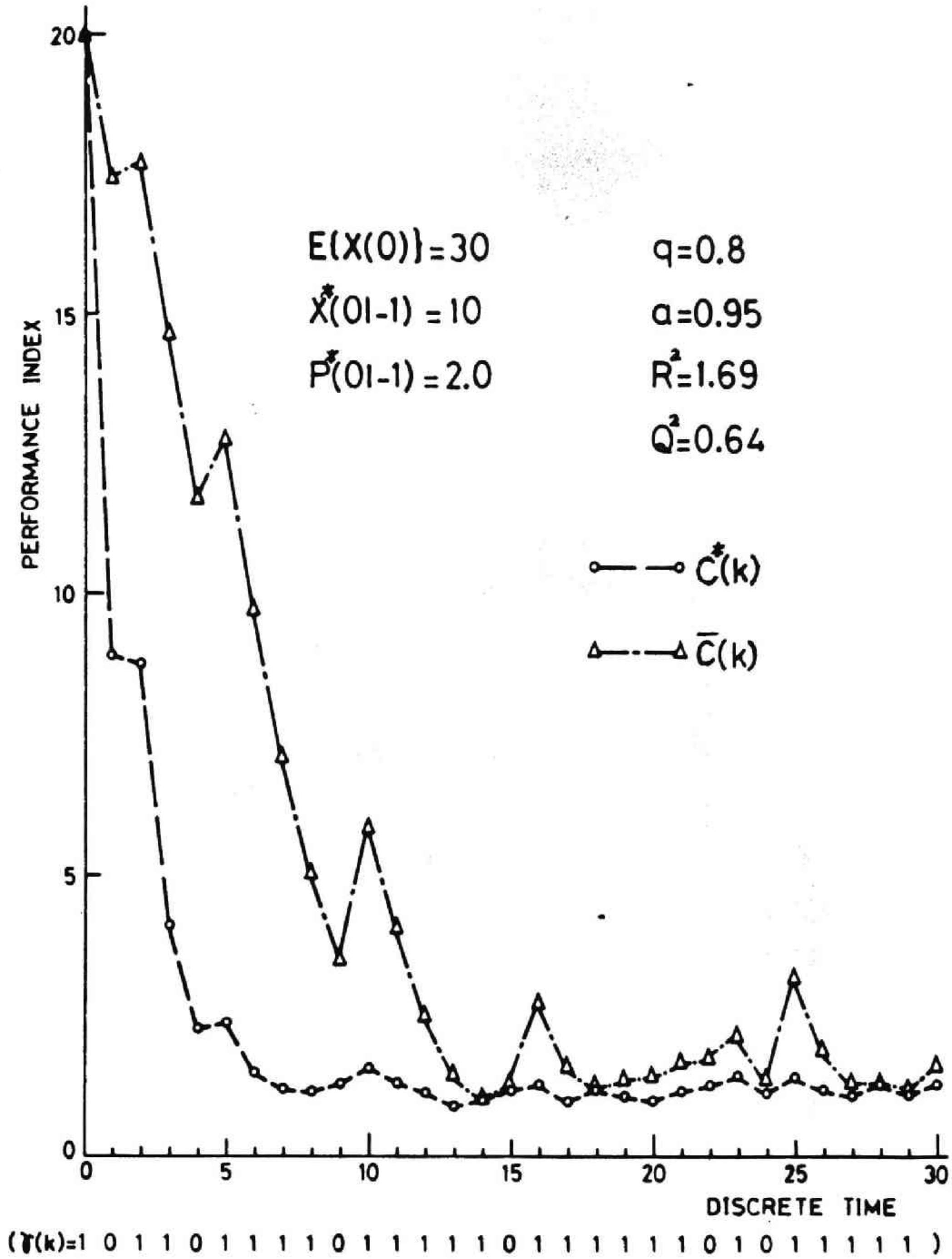

Fig. 2.4. Performance indices $C^*(k)$ and $\overline{C}(k)$ for 50 run average, where the realization of $\gamma(k)$'s is fixed.

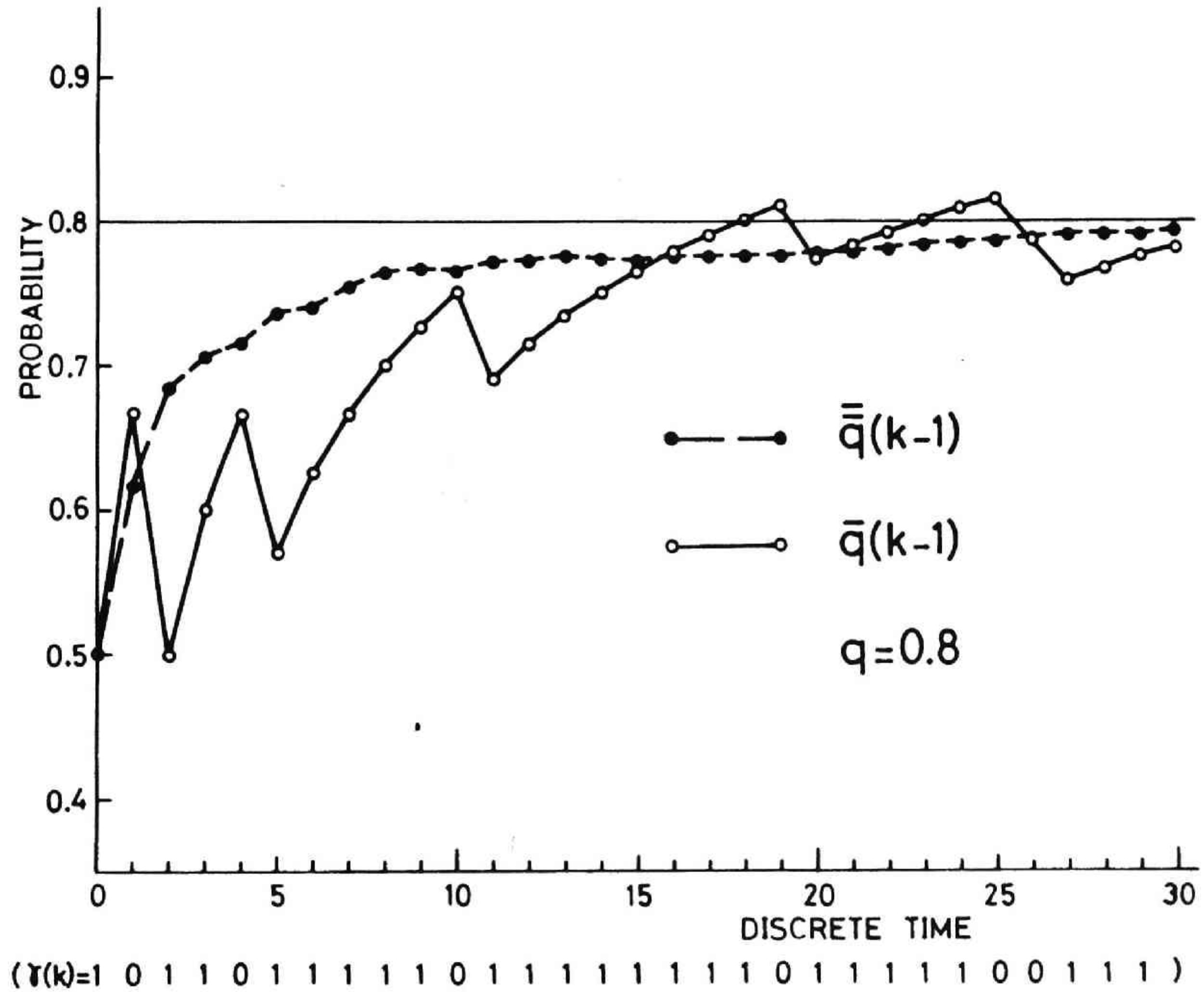

Fig. 2.5. Sample path $\bar{q}(k-1)$ corresponding to the realization of $\gamma(k)$'s indicated, and 50 run average $\bar{\bar{q}}(k-1)$ of $\bar{q}(k-1)$.

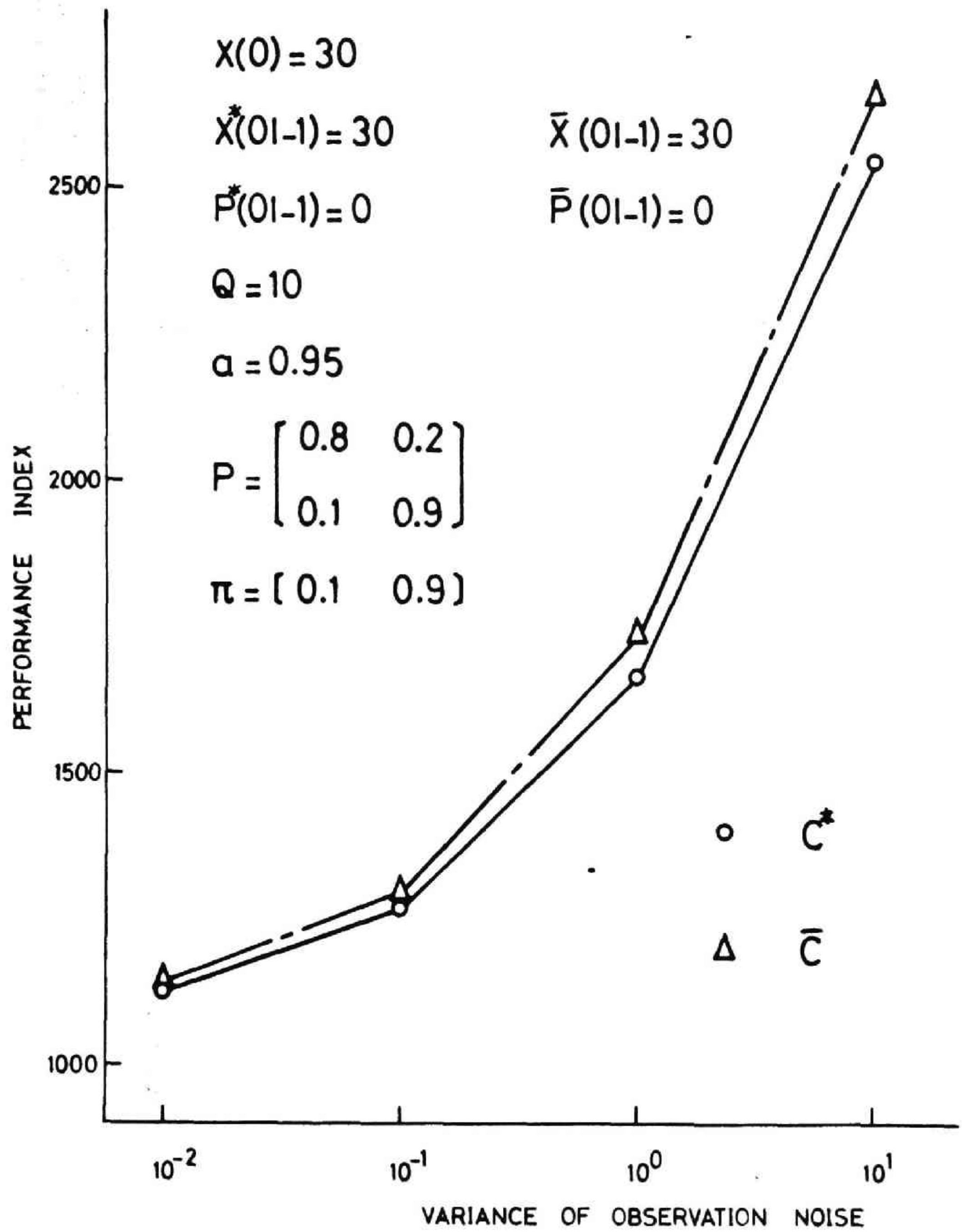

Fig. 2.6. Performance indices C* and $\overline{C}$ with respect to the value of the variance of the observation noise.

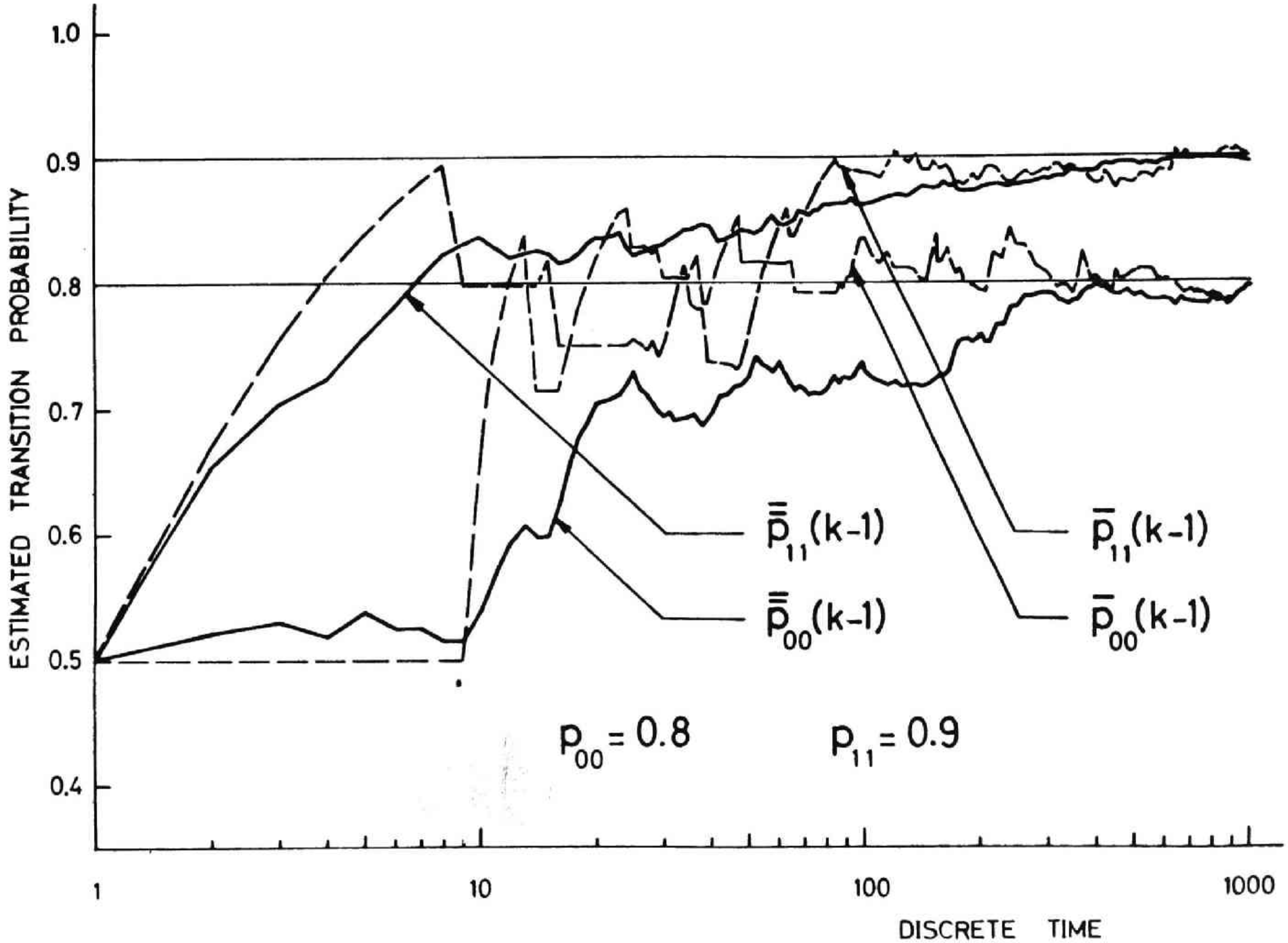

Fig. 2.7. Sample paths $\bar{p}_{ii}(k-1)$ and 10 run averages $\bar{\bar{p}}_{ii}(k-1)$ of $\bar{p}_{ii}(k-1)$ $(i=0,1)$, where $R^2 = 1.69$.

CHAPTER III

STATE ESTIMATION FOR LINEAR CONTINUOUS SYSTEM

WITH JUMP PROCESS

3.1 Introduction

Chapter 3 is devoted to the derivation of the optimal and suboptimal estimator algorithms for a class of linear continuous systems modulated by a multi-valued jump process, which are the continuous-time counterpart of those treated in chapter 2. The approach adopted here is as follows. First, we express the jump process in terms of its initial value, the jump times and the values taken by the jump process after the jump, and then we apply general Bayes' rule (Kallianpur and Striebel, 1968) and the likelihood-ratio formula (Duncan, 1968; Kailath, 1969; Wong, 1971; Lipčer and Širjaev, 1972) to obtain the a posteriori probability distribution of the jump process. The minimum-variance estimate is given in terms of the a posteriori probability distribution of the jump process and the Kalman-filter estimates corresponding to the admissible values of the jump process. Another approach based upon Lainiotis' formula (Lainiotis, 1971) is also taken, which is useful for deriving approximate estimator algorithms.

In section 3.2, we precisely formulate the estimation problem for continuous systems with a jump Markov process. Section 3.3 is devoted to the derivation of the minimum-variance estimator algorithm. The optimal estimator algorithm is, however, infinite dimensional, so that feasible approximate estimator algorithms are presented in section 3.4. In section 3.5, treated are the special cases of (i) linear continuous systems with system-component failure and (ii) linear continuous systems with interrupted observation mechanisms. In section 3.6, simulation studies are carried out to illustrate the behavior of the optimal estimator and to demonstrate the feasibilty of the proposed approximate estimator algorithms.

3.2 Statement of Problem

Consider the system represented by a stochastic differential equation

$$dx(t) = F(t,\gamma(t))x(t)dt + G(t,\gamma(t))u(t)dt + Q(t,\gamma(t))dw(t), \quad t \geq 0 , \tag{2.1}$$

and let the observation be given by

$$dy(t) = H(t,\gamma(t))x(t)dt + R(t)dv(t), \quad t \geq 0 , \tag{2.2}$$

where

$x(t)$: an $n \times 1$ state vector at time t;

$y(t)$: a $p \times 1$ observation vector;

$u(t)$: a $q \times 1$ deterministic input vector;

$F(\cdot,\cdot)$: an $n \times n$ state transition matrix;

$G(\cdot,\cdot)$: an $n \times q$ gain matrix;

$H(\cdot,\cdot)$: a $p \times n$ observation matrix;

$Q(\cdot,\cdot)$: an $n \times m$ matrix;

$R(\cdot)$: a $p \times p$ nonsingular matrix;

$w(t)$: an $m \times 1$ standard Wiener process with unit variance matrix;

$v(t)$: a $p \times 1$ standard Wiener process with unit variance matrix;

and

$\gamma(t)$: a right-continuous jump Markov process taking on values of $1, 2, \ldots, M$ where M is a positive integer.

We assume that the initial state $x(0)$ is Gaussian with mean $\hat{x}(0|0) \triangleq E\{x(0)\}$ and covariance $\hat{P}(0|0) \triangleq Cov\{x(0)\}$. It is also assumed that $F(t,\cdot)$, $G(t,\cdot)$, $H(t,\cdot)$, $Q(t,\cdot)$ and $R(t)$ are continuous in t and that the stochastic processes $\gamma(t)$, $w(t)$, $v(t)$ and $x(0)$ are mutually independent.

The main objective in this chapter is to find the minimum-variance estimate $\hat{x}(t|t)$ by observing the data $\{y(s), 0 \leqq s < t\}$. It is well known that the minimum-variance estimate $\hat{x}(t|t)$ is given by the conditional expectation

$$\hat{x}(t|t) = E\{x(t)|Y^t\}, \tag{2.3}$$

where Y^t denotes the observation record $\{y(s), 0 \leqq s < t\}$.

Let us define:

$$\tau_k = \text{the random time that the kth jump of the process } \gamma(t) \text{ occurs;} \tag{2.4}$$

$$j_k = \text{the random value taken by the jump process after the kth jump, that is, } \gamma(\tau_k+0); \tag{2.5}$$

$$q_{i_{k-1},i_k}(t) = \lim_{s \downarrow t} \frac{P(\gamma(s)=i_k|\gamma(t)=i_{k-1})}{s - t}; \tag{2.6}$$

and

$$q_{i_{k-1}}(t) = \sum_{\substack{i_k=1 \\ i_k \neq i_{k-1}}}^{M} q_{i_{k-1},i_k}(t), \quad k = 1, 2, \ldots, \tag{2.7}$$

where we assume that the limits (2.6) exist uniformly in t and that the functions $q_{i_{k-1},i_k}(t)$ are continuous in t (Gikhman and Skorokhod, 1969). It is also assumed that the inequality:

$$\int_0^t E\left\{\|H(s,\gamma(s))x(s)\|^2_{\{R(s)R'(s)\}^{-1}}\right\}ds < \infty \qquad (2.8)$$

holds for any $t > 0$, where for any vector a and matrix A of appropriate dimensions,

$$\|a\|^2_A = a'Aa \ .$$

3.3 Optimal Estimator Algorithm

We shall first show the optimal estimator algorithm and then prove it by two different approaches.

Let us define $a^n \triangleq (a_1,a_2,\ldots,a_n)'$ for any sequence $\{a_k,\ k = 1,\ 2,\ldots\}$. Then the minimum-variance estimator algorithm for system (2.1) and (2.2) is given by the following.

Theorem 3.1 (Optimal Estimator Algorithm)

For the system described by (2.1) and (2.2), the minimum-variance estimate $\hat{x}(t|t)$ is given by

$$\hat{x}(t|t) = \sum_{i_0=1}^{M} \hat{x}(t|\gamma(0)=i_0,\tau_1 \geq t)P(\gamma(0)=i_0,\tau_1 \geq t|Y^t) +$$

$$+ \sum_{n=1}^{\infty} \sum_{i_0=1}^{M} \sum_{\substack{i_1=1 \\ i_1 \neq i_0}}^{M} \cdots \sum_{\substack{i_n=1 \\ i_n \neq i_{n-1}}}^{M} \int_0^t \int_{t_1}^t \cdots \int_{t_{n-1}}^t \hat{x}(t|\gamma(0)=i_0, \tau^n=t^n, j^n=i^n, \tau_{n+1} \geq t)$$

$$\times P(\gamma(0)=i_0, \tau^n \varepsilon dt^n, j^n=i^n, \tau_{n+1} \geq t | Y^t), \quad (3.1)$$

where $\hat{x}(t|\gamma(0)=i_0, \tau_1 \geq t)$ and $\hat{x}(t|\gamma(0)=i_0, \tau^n=t^n, j^n=i^n, \tau_{n+1} \geq t)$ are defined by

$$\hat{x}(t|\gamma(0)=i_0, \tau_1 \geq t) \triangleq E\{x(t)|\gamma(0)=i_0, \tau_1 \geq t, Y^t\} \quad (3.2)_1$$

and

$$\hat{x}(t|\gamma(0)=i_0, \tau^n=t^n, j^n=i^n, \tau_{n+1} \geq t)$$

$$\triangleq E\{x(t)|\gamma(0)=i_0, \tau^n=t^n, j^n=i^n, \tau_{n+1} \geq t, Y^t\} \quad (3.2)_2$$

and are given by the usual Kalman-filter algorithms using the values of $\{\gamma(s), 0 \leq s < t\}$ specified as conditioning. The a posteriori probabilities $P(\gamma(0)=i_0, \tau_1 \geq t|Y^t)$ and $P(\gamma(0)=i_0, \tau^n \varepsilon dt^n, j^n=i^n, \tau_{n+1} \geq t|Y^t)$ are given by

$$P(\gamma(0)=i_0, \tau_1 \geq t|Y^t) = \frac{L(t, \{\gamma(0)=i_0, \tau_1 \geq t\}) P(\gamma(0)=i_0, \tau_1 \geq t)}{L(t)} \quad (3.3)$$

and

$$P(\gamma(0)=i_0, \tau^n \varepsilon dt^n, j^n=i^n, \tau_{n+1} \geq t|Y^t)$$

$$= \frac{L(t,\{\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}\geq t\})P(\gamma(0)=i_0,\tau^n \varepsilon dt^n,j^n=i^n,\tau_{n+1}\geq t)}{L(t)} \quad (3.4)$$

where

$$L(t,\{\gamma(s),0\leq s<t\})$$

$$= \exp\left\{\int_0^t \hat{x}'(s|\gamma(u),0\leq u<s)H'(s,\gamma(s))\{R(s)R'(s)\}^{-1}dy(s) - \frac{1}{2}\int_0^t \|H(s,\gamma(s))\hat{x}(s|\gamma(u),0\leq u<s)\|^2_{\{R(s)R'(s)\}^{-1}}\, ds\right\} \quad (3.5)$$

$$L(t) = \sum_{i_0=1}^{M} L(t,\{\gamma(0)=i_0,\tau_1\geq t\})P(\gamma(0)=i_0,\tau_1\geq t|Y^t)$$
$$+ \sum_{n=1}^{\infty}\sum_{i_0=1}^{M}\sum_{\substack{i_1=1\\ i_1\neq i_0}}^{M}\cdots\sum_{\substack{i_n=1\\ i_n\neq i_{n-1}}}^{M}\int_0^t\int_{t_1}^t\cdots\int_{t_{n-1}}^t L(t,\{\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}\geq t\})$$
$$\times P(\gamma(0)=i_0,\tau^n \varepsilon dt^n,j^n=i^n,\tau_{n+1}\geq t) \quad (3.6)$$

and

$$\hat{x}(s|\gamma(u),0\leq u<s) = E\{x(s)|\gamma(u),0\leq u<s,Y^s\}\ .$$

Here, the likelihood ratios $L(t,\{\gamma(0)=i_0,\tau_1\geq t\})$ and $L(t,\{\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}\geq t\})$ are obtained by substituting the respective values of $\gamma(s)$ $(0\leq s<t)$ specified in the brackets into (3.5). Moreover, the a priori probabilities $P(\gamma(0)=i_0,\tau_1\geq t)$ and $P(\gamma(0)=i_0,\tau^n \varepsilon dt^n,j^n=i^n,\tau_{n+1}\geq t)$ appearing in (3.3) and (3.4) are

expressed as

$$P(\gamma(0)=i_0,\tau_1\geqq t) = P(\gamma(0)=i_0)\exp\left\{ - \int_0^t q_{i_0}(s)ds\right\} \tag{3.7}$$

and

$$\begin{aligned} &P(\gamma(0)=i_0,\tau^n \varepsilon dt^n, j^n=i^n, \tau_{n+1}\geqq t) \\ &= P(\gamma(0)=i_0)\exp\left\{-\int_{t_n}^{t} q_{i_n}(s)ds - \int_{t_{n-1}}^{t_n} q_{i_{n-1}}(s)ds - \cdots \right. \\ &\qquad \left. \cdots - \int_0^{t_1} q_{i_0}(s)ds\right\} \\ &\quad \times q_{i_{n-1},i_n}(t_n)\cdots q_{i_0,i_1}(t_1)dt_n\cdots dt_1 , \end{aligned} \tag{3.8}$$

where $q_{i_{k-1}}(t)$ and $q_{i_{k-1},i_k}(t)$ $(k=1,2,\ldots)$ are defined by (2.6) and (2.7).

Remark 1 : It should be noted that the optimal estimator algorithm for continuous systems can be obtained as the limiting case of the optimal algorithm for discrete systems presented in section 2.4, chapter II. However, we can not obtain the optimal algorithm as the limiting case of the discrete one presented in section 2.3, chapter II. The success in obtaining the present theorem is mainly due to the fact that we express the jump process $\gamma(t)$ in terms of the random sequence $\{\gamma(0),\tau_1,j_1,\tau_2,j_2,\ldots\}$, which has been already used in section 2.4, chapter II.

Remark 2 : $\hat{x}(t|\gamma(0)=i_0,\tau_1\geqq t)$ and $\hat{x}(t|\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}\geqq t)$ defined by (3.2) can be obtained by the usual Kalman-filter algorithms, because the behavior of the random process $\{\gamma(s),0\leqq s<t\}$ is specified as conditioning. Therefore, equation (3.1) means that the minimum-variance estimate $\hat{x}(t|t)$ is given by averaging the conditional Kalman-filter estimates given the values of the jump process $\{\gamma(s),0\leqq s<t\}$. The a posteriori probability distribution of the jump process is given by (3.3) and (3.4).

Remark 3 : It should be noted that since the right-hand side of (3.1) consists of infinitely many terms, the evaluation of the minimum-variance estimate $\hat{x}(t|t)$ through (3.1) is not feasible except for the case when, for some finite positive integer N, the jump process $\{\gamma(s),0\leqq s<t\}$ does not jump more than N times with probability 1. Thus the feasible approximate minimum-variance estimator algorithms are required for the practical implementation, which will be discussed in the next section.

Remark 4 : It should also be noted that the value of the likelihood ratio $L(t)$ given by (3.6) is finite because of assumption (2.8).

We shall present two different proofs of the above theorem. Of the two, the former is compact and straightforward; and the latter is somewhat involved but useful for deriving approximate estimator algorithms which will be shown in the next section.

A Proof of Theorem 3.1

By definition (2.4) and (2.5) and the smoothing property of conditional expectations (Doob, 1953), the conditional expectation (2.3) can be expressed as

$$\hat{x}(t|t) = E\{E\{x(t)|\gamma(0),\tau_1,j_1,\tau_2,j_2,\dots,Y^t\}|Y^t\}$$

$$= \sum_{i_0=1}^{M} E\{x(t)|\gamma(0)=i_0,\tau_1 \geq t,Y^t\}P(\gamma(0)=i_0,\tau_1 \geq t|Y^t)$$

$$+ \sum_{n=1}^{\infty} \sum_{i_0=1}^{M} \sum_{\substack{i_1=1 \\ i_1 \neq i_0}}^{M} \cdots \sum_{\substack{i_n=1 \\ i_n \neq i_{n-1}}}^{M} \int_0^t \int_{t_1}^t \cdots \int_{t_{n-1}}^t E\{x(t)|\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1} \geq t,Y^t\}$$

$$\times P(\gamma(0)=i_0,\tau^n \epsilon dt^n,j^n=i^n,\tau_{n+1} \geq t|Y^t) \ . \qquad (3.9)$$

From the above equation (3.9), we obtain (3.1); the conditional expectations $E\{x(t)|\gamma(0)=i_0,\tau_1 \geq t,Y^t\}$ and $E\{x(t)|\gamma(0)=i_0,\tau^n=t^n, j^n=i^n,\tau_{n+1} \geq t,Y^t\}$ appearing in (3.9) can be obtained by the usual Kalman-filter algorithms, because the behavior of the jump process $\{\gamma(s),0 \leq s<t\}$ is specified as conditioning. Therefore, in order to complete the proof of the theorem, it is sufficient to derive the algorithms that produce the a posteriori probability distribution (3.3) and (3.4) of the jump process.

Suppose that P^1 and P^0 are two probability measures on continuous Y^t function space which generate the process $\{y(s), 0 \leq s < t\}$ under the hypotheses

$$h_1 : \ dy(s) = H(s,\gamma(s))x(s)ds + R(s)dv(s), \qquad \text{for signal present,} \qquad (3.10)_1$$

and

$$h_0 : \ dy(s) = R(s)dv(s), \quad \text{for signal not present,} \qquad (3.10)_2$$

respectively. Also, suppose $P^1_{\{\gamma(s), 0 \leq s < t\}}$ is the measure which generates the process $\{y(s), 0 \leq s < t\}$ under the conditions that the signal is present and that the values of $\gamma(s)$ $(0 \leq s < t)$ in the bracket are fixed. Since for any Borel set A of Y^t function space we have the equality

$$P^1(A) = E\{P^1_{\{\gamma(s), 0 \leq s < t\}}(A)\}, \qquad (3.11)$$

the measure $P^1_{\{\gamma(s), 0 \leq s < t\}}$ is absolutely continuous with respect to measure P^1 with probability 1. Thus there exists Radon-Nikodym derivative $dP^1_{\{\gamma(0)=i_0, \tau_1 \geq t\}} / dP^1$ with probability 1. Therefore, by the general Bayes' rule (Kallianpur and Striebel, 1968), we obtain

$$P(\gamma(0)=i_0, \tau_1 \geq t | Y^t) = \frac{dP^1_{\{\gamma(0)=i_0, \tau_1 \geq t\}}}{dP^1}(Y^t) P(\gamma(0)=i_0, \tau_1 \geq t) \qquad (3.12)$$

with probability 1. Also, by the property of Radon-Nikodym derivatives (Wong, 1971; Loeve, 1963), we get

$$\frac{dP^1_{\{\gamma(0)=i_0,\tau_1\geq t\}}}{dP^1} = \frac{dP^1_{\{\gamma(0)=i_0,\tau_1\geq t\}}}{dP^0} \Big/ \frac{dP^1}{dP^0} \tag{3.13}$$

with P^1-measure 1, where the Radon-Nikodym derivatives

$$\frac{dP^1_{\{\gamma(0)=i_0,\tau_1\geq t\}}}{dP^0} \quad \text{and} \quad \frac{dP^1}{dP^0}$$

exist from assumption (2.8). Therefore, from (3.12) and (3.13) we obtain (3.3), where

$$L(t,\{\gamma(0)=i_0,\tau_1\geq t\}) \triangleq \frac{dP^1_{\{\gamma(0)=i_0,\tau_1\geq t\}}}{dP^0} (Y^t) \tag{3.14}$$

and

$$L(t) \triangleq \frac{dP^1}{dP^0} (Y^t) \ . \tag{3.15}$$

By the same procedure used in deriving (3.3), we obtain (3.4), where

$$L(t,\{\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}\geq t\}) \triangleq \frac{dP^1_{\{\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}\geq t\}}}{dP^0} (Y^t) \ . \tag{3.16}$$

Likelihood ratio $L(t,\{\gamma(s),0\leqq s<t\}) = dP^1_{\{\gamma(s),0\leqq s<t\}}/dP^0(Y^t)$ is given by (3.5) (Duncan, 1968; Kailath, 1969; Wong, 1971; Lipčer and Širjaev, 1972), because the behavior of the jump process $\{\gamma(s),0\leqq s<t\}$ is specified. The likelihood ratios (3.14) and (3.16) are obtained by substituting the respective values of $\gamma(s)$ $(0\leqq s<t)$ specified in the brackets into (3.5). Also, the likelihood ratio (3.15) is given by

$$L(t) = E_{Y^t}\left\{\frac{dP^1_{\{\gamma(s),0\leqq s<t\}}}{dP^0}(Y^t)\right\}, \tag{3.17}$$

where $E_{Y^t}\{\cdot\}$ denotes the expectation over $\{\gamma(s),0\leqq s<t\}$ with Y^t fixed. From (3.17), likelihood ratio $L(t)$ is expressed as (3.6). Moreover, the probabilities $P(\gamma(0)=i_0,\tau_1\geqq t)$ and $P(\gamma(0)=i_0,\tau^n\epsilon dt^n,j^n=i^n,\tau_{n+1}\geqq t)$ are expressed as

$$P(\gamma(0)=i_0,\tau_1\geqq t) = P(\gamma(0)=i_0)P(\tau_1\geqq t|\gamma(0)=i_0) \tag{3.18}$$

and

$$\begin{aligned}
&P(\gamma(0)=i_0,\tau^n\epsilon dt^n,j^n=i^n,\tau_{n+1}\geqq t)\\
&= P(\tau_{n+1}\geqq t|\tau_n=t_n,j_n=i_n)P(j_n=i_n|\tau_n=t_n,j_{n-1}=i_{n-1})\\
&\quad\times p(\tau_n=t_n|\tau_{n-1}=t_{n-1},j_{n-1}=i_{n-1})dt_n \cdots\\
&\quad\times P(j_1=i_1|\tau_1=t_1,\gamma(0)=i_0)p(\tau_1=t_1|\gamma(0)=i_0)dt_1P(\gamma(0)=i_0).
\end{aligned} \tag{3.19}$$

From (3.18) and (3.19), we can easily obtain (3.7) and (3.8) (Gikhman and Skorokhod, 1969). This completes the proof of the theorem.

Next, we shall show another proof based upon Lainiotis' formula (Lainiotis, 1971).

Another Proof of Theorem 3.1

We derive (3.3)-(3.6) by applying Lainiotis' formula. We assume that there is an underlying probability space (Ω,β,P), where Ω is an abstract set consisting of elementary events ω, β is a σ-field of subsets of Ω and P is a probability measure defined on β. Now, define the number of jumps which occur in the time interval $[0,t)$ as

$$n_\tau^t(\omega) \triangleq \sup\{n \,|\, \tau_n(\omega) \in [0,t)\},$$

and the measurable subset B_N^t of Ω as

$$B_N^t \triangleq \{\omega \,|\, n_\tau^t(\omega) \leqq N\}, \tag{3.20}$$

where N is a positive integer; that is, if $\omega \in B_N^t$, $\{\gamma(s,\omega),\ 0 \leqq s < t\}$ does not jump more than N times. Therefore, if we confine the elementary events ω in $B_N^t \subset \Omega$, we can regard the initial value, jump times and the values taken after the jump up to time t as the unknown $2N+1$-dimensional parameter vector for linear system (2.1) and (2.2) in the time interval $[0,t)$. Thus, by applying Lainiotis' formula (see Appendix I), we can obtain the following.

Lemma 1

$$P(\gamma(0)=i_0,\tau_1 \geqq t \mid B_N^t, Y^t)$$
$$= \frac{L(t,\{\gamma(0)=i_0,\tau_1 \geqq t\})P(\gamma(0)=i_0,\tau_1 \geqq t \mid B_N^t)}{L_N(t)} \tag{3.21}$$

$$P(\gamma(0)=i_0,\tau^n \varepsilon dt^n, j^n=i^n, \tau_{n+1} \geqq t \mid B_N^t, Y^t)$$
$$= \frac{L(t,\{\gamma(0)=i_0,\tau^n=t^n, j^n=i^n, \tau_{n+1} \geqq t\})P(\gamma(0)=i_0,\tau^n \varepsilon dt^n, j^n=i^n, \tau_{n+1} \geqq t \mid B_N^t)}{L_N(t)} \tag{3.22}$$

where $0 < t_1 < t_2 < \cdots < t_n < t$, $n = 1, 2, \ldots, N$, $i_0 = 1, 2, \ldots, M$, $L(t,\{\gamma(0)=i_0,\tau_1 \geqq t\})$ and $L(t,\{\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1} \geqq t\})$ are given by substituting the values of $\gamma(s)$ $(0 \leqq s < t)$ specified in the brackets into (3.5), and $L_N(t)$ is defined by

$$L_N(t) = \sum_{i_0=1}^{M} L(t,\{\gamma(0)=i_0,\tau_1 \geqq t\})P(\gamma(0)=i_0,\tau_1 \geqq t \mid B_N^t)$$
$$+ \sum_{n=1}^{N} \sum_{i_0=1}^{M} \sum_{\substack{i_1=1 \\ i_1 \neq i_0}}^{M} \cdots \sum_{\substack{i_n=1 \\ i_n \neq i_{n-1}}}^{M} \int_0^t \int_{t_1}^t \cdots \int_{t_{n-1}}^t L(t,\{\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1} \geqq t\})$$
$$\times P(\gamma(0)=i_0,\tau^n \varepsilon dt^n, j^n=i^n, \tau_{n+1} \geqq t \mid B_N^t) \tag{3.23}$$

From Lemma 1, we can easily obtain the following lemma (see Appendix II).

Lemma 2

$$P(\gamma(0)=i_0,\tau_1\geqq t|B_N^t,Y^t)$$

$$=\frac{L(t,\{\gamma(0)=i_0,\tau_1\geqq t\})P(\gamma(0)=i_0,\tau_1\geqq t)}{L_N'(t)} \tag{3.24}$$

$$P(\gamma(0)=i_0,\tau^n\varepsilon dt^n,j^n=i^n,\tau_{n+1}\geqq t|B_N^t,Y^t)$$

$$=\frac{L(t,\{\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}\geqq t\})P(\gamma(0)=i_0,\tau^n\varepsilon dt^n,j^n=i^n,\tau_{n+1}\geqq t)}{L_N'(t)} \tag{3.25}$$

where $0 < t_1 < t_2 < \cdots < t_n < t$, $n = 1, 2,\ldots, N$, $i_0 = 1, 2,\ldots$ $\ldots, M$ and

$$L_N'(t) = \sum_{i_0=1}^{M} L(t,\{\gamma(0)=i_0,\tau_1\geqq t\})P(\gamma(0)=i_0,\tau_1\geqq t)$$
$$+\sum_{n=1}^{N}\sum_{i_0=1}^{M}\sum_{\substack{i_1=1\\ i_1\neq i_0}}^{M}\cdots\sum_{\substack{i_n=1\\ i_n\neq i_{n-1}}}^{M}\int_0^t\int_{t_1}^t\cdots\int_{t_{n-1}}^t L(t,\{\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}\geqq t\})$$
$$\times P(\gamma(0)=i_0,\tau^n\varepsilon dt^n,j^n=i^n,\tau_{n+1}\geqq t). \tag{3.26}$$

Noting that for every finite $t > 0$

$$\lim_{N\to\infty} B_N^t = \Omega - \Lambda^t$$

holds, where $P(\Lambda^t) = 0$ (Gikhman and Skorokhod, 1969), and that $L_N'(t)$ is positive and monotone increasing as N, we can easily show that both sides of (3.24) and (3.25) converge to (3.3) and (3.4), respectively, as $N \to \infty$. Also, (3.26) converges to (3.6) as $N \to \infty$. This completes the proof of Theorem 3.1.

Remark 5 : The result of Lemma 2 is useful for obtaining approximate estimator algorithms which will be shown in the next section. Also, note that equations (3.1)-(3.6) are the continuous-time analog of (4.2)-(4.9) in chapter II.

3.4 Approximate Estimator Algorithms

In this section, based upon Lemma 2 in the previous section, we shall present approximate estimator algorithms for two cases: (i) the time interval of operation is finite and fixed, and (ii) the time interval of operation is free and may be infinite.

3.4.1 Finite Interval of Operation

Let T be the fixed final time. Then for a sufficiently small positive number ε, define N_0 as a positive integer such that if $N \geqq N_0$ the inequality

$$P(B_N^T) > 1 - \varepsilon \tag{4.1}$$

is satisfied. In other words, the jump process $\gamma(t)$ jumps more than N_0 times in the time interval $[0,T)$ with probability less than ε. Therefore, we can obtain the optimal estimate of $x(t)$ with probability greater than $1 - \varepsilon$, if we take

$$x^*(t|t) \triangleq E\{x(t)|Y^t, B_{N_0}^t\} \tag{4.2}$$

as the estimate of $x(t)$. By the smoothing property of conditional expectations, (4.2) is expressed as

$$\begin{aligned}
x^*(t|t) &= E\{E\{x(t)|\gamma(0), \tau^{N_0}, j^{N_0}, Y^t, B_{N_0}^t\}|Y^t, B_{N_0}^t\} \\
&= \sum_{i_0=1}^{M} \hat{x}(t|\gamma(0)=i_0, \tau_1 \geqq t) P(\gamma(0)=i_0, \tau_1 \geqq t | B_{N_0}^t, Y^t) \\
&+ \sum_{n=1}^{N_0} \sum_{i_0=1}^{M} \sum_{\substack{i_1=1 \\ i_1 \neq i_0}}^{M} \cdots \sum_{\substack{i_n=1 \\ i_n \neq i_{n-1}}}^{M} \int_0^t \int_{t_1}^t \cdots \int_{t_{n-1}}^t \hat{x}(t|\gamma(0)=i_0, \tau^n=t^n, j^n=i^n, \tau_{n+1} \geqq t) \\
&\qquad \times P(\gamma(0)=i_0, \tau^n \varepsilon dt^n, j^n=i^n, \tau_{n+1} \geqq t | B_{N_0}^t, Y^t),
\end{aligned} \tag{4.3}$$

where $\hat{x}(t|\gamma(0)=i_0, \tau_1 \geqq t)$ and $\hat{x}(t|\gamma(0)=i_0, \tau^n=t^n, j^n=i^n, \tau_{n+1} \geqq t)$ are defined by (3.2). Equation (4.3) together with Lemma 2 gives the approximate estimate $x^*(t|t)$.

It should be noted that the positive number ε is related to the desired accuracy of the approximate estimate and that if $\varepsilon \to +0$, the approximate estimate given by (4.3) converges to the optimal estimate given by (3.1) with probability 1. It should, however, be noted that we can not make the approximate estimate arbitrarily close to the optimal one. For, $N \to \infty$ as $\varepsilon \to +0$ and then summation of infinitely many terms must be carried out.

3.4.2 Infinite Interval of Operation

In order to simplify the following discussion, we assume that the jump process $\gamma(t)$ is stationary. Let N_0 be a positive integer determined a priori from the available amount of computer memory, and for a sufficiently small positive number ε, define t_0 as a positive number such that if $t \leqq t_0$ the inequality

$$\min_{i_0 \varepsilon \{1,2,\ldots,M\}} P(B_{N_0}^{t} | \gamma(0)=i_0) > 1 - \varepsilon \tag{4.4}$$

is satisfied. This means that with probability greater than $1 - \varepsilon$ the jump process $\gamma(t)$ does not jump more than N_0 times in the interval $[0,t_0)$ regardless of the values of $\gamma(0)$. Let us define

$\bar{\tau}_n^t$ = the random time that the nth jump of the process $\gamma(t)$ occurs after the time $t - t_0$;

$$\bar{\tau}_t^n = (\bar{\tau}_1^t, \bar{\tau}_2^t, \ldots, \bar{\tau}_n^t);$$

$$\bar{j}_n^t = \gamma(\bar{\tau}_n^t + 0);$$

$$\bar{j}_t^n = (\bar{j}_1^t, \bar{j}_2^t, \ldots, \bar{j}_n^t);$$

$$\bar{n}_\tau^t = \sup\{n \mid \bar{\tau}_n^t \in [t-t_0, t)\} \;;$$

and

$$\bar{B}_{N_0}^t = \{\omega \mid \bar{n}_\tau^t(\omega) \leqq N_0\} \;.$$

By the definition of t_0 and the assumption of stationarity of the jump process $\gamma(t)$,

$$P(\bar{B}_{N_0}^t) = \sum_{i_0=1}^{M} P(\bar{B}_{N_0}^t \mid \gamma(t-t_0)=i_0) P(\gamma(t-t_0)=i_0) > 1 - \varepsilon. \tag{4.5}$$

We shall present a feasible approximate estimator algorithm under the following

Assumption : the conditional probability density function $p(x(t-t_0) \mid Y^{t-t_0})$ is Gaussian with

$$\text{mean} = x^*(t-t_0 \mid t-t_0) \triangleq E\{x(t-t_0) \mid Y^{t-t_0}\} \qquad (4.6)_1$$

$$\text{cov} = P^*(t-t_0 \mid t-t_0) \triangleq \text{Cov}\{x(t-t_0) \mid Y^{t-t_0}\} \qquad (4.6)_2$$

for any time t ($> t_0$).

Based upon assumption (4.6), we can easily see from (4.5) that if we take $t - t_0$ as the initial time and t as the final time T, the integer N_0 also satisfies inequality (4.1) and the approximate estimator algorithm presented for the case of the finite interval of operation treated in the previous subsection can be applied to this case. Thus, for $t > t_0$, we obtain the approximate estimate

$$x^*(t|t) = \sum_{i_0=1}^{M} \hat{x}(t|\gamma(t-t_0)=i_0,\bar{\tau}_1^t \geq t)P(\gamma(t-t_0)=i_0,\bar{\tau}_1^t \geq t|\bar{B}_{N_0}^t,Y^t)$$

$$+ \sum_{n=1}^{N_0} \sum_{i_0=1}^{M} \cdots \sum_{\substack{i_n=1\\ i_n \neq i_{n-1}}}^{M} \int_{t-t_0}^{t} \int_{t_1}^{t} \cdots \int_{t_{n-1}}^{t} \hat{x}(t|\gamma(t-t_0)=i_0,\bar{\tau}_t^n=t^n,\bar{j}_t^n=i^n,\bar{\tau}_{n+1}^t \geq t)$$

$$\times P(\gamma(t-t_0)=i_0,\bar{\tau}_t^n \epsilon dt^n,\bar{j}_t^n=i^n,\bar{\tau}_{n+1}^t \geq t|\bar{B}_{N_0}^t,Y^t), \qquad (4.7)$$

where

$$\hat{x}(t|\gamma(t-t_0)=i_0,\bar{\tau}_1^t \geq t) \triangleq E\{x(t)|I^*(t-t_0),Y^t,\gamma(t-t_0)=i_0,\bar{\tau}_1^t \geq t\} \qquad (4.8)_1$$

$$\hat{x}(t|\gamma(t-t_0)=i_0,\bar{\tau}_t^n=t^n,\bar{j}_t^n=i^n,\bar{\tau}_{n+1}^t \geq t)$$

$$\triangleq E\{x(t)|I^*(t-t_0),Y^t,\gamma(t-t_0)=i_0,\bar{\tau}_t^n=t^n,\bar{j}_t^n=i^n,\bar{\tau}_{n+1}^t \geq t\} \qquad (4.8)_2$$

and

$$I^*(t-t_0) \triangleq \{x^*(t-t_0|t-t_0),P^*(t-t_0|t-t_0)\}.$$

Here, from assumption (4.6), $\hat{x}(t|\gamma(t-t_0)=i_0,\bar{\tau}_1^t \geq t)$ and $\hat{x}(t|\gamma(t-t_0)=i_0,\bar{\tau}_t^n=t^n,\bar{j}_t^n=i^n,\bar{\tau}_{n+1}^t \geq t)$ are obtained by the usual Kalman-filter algorithms assuming that the initial time is $t - t_0$ and the a priori information is $x^*(t-t_0|t-t_0)$ and $P^*(t-t_0|t-t_0)$. Also, from (4.7) and assumption (4.6), the associated covariance matrix $P^*(t|t) \triangleq \mathrm{Cov}\{x(t)|Y^t\}$ is approximately given by

$$
\begin{aligned}
P^*(t|t) &= \mathrm{Cov}\{x(t)|I^*(t-t_0),Y^t\} \\
&= E\Big\{\mathrm{Cov}\{x(t)|I^*(t-t_0),\gamma(t-t_0),\overline{\tau}_1^t,\overline{j}_1^t,\overline{\tau}_2^t,\overline{j}_2^t,\ldots,Y^t\} \\
&\quad + [x^*(t|t) - E\{x(t)|I^*(t-t_0),\gamma(t-t_0),\overline{\tau}_1^t,\overline{j}_1^t,\overline{\tau}_2^t,\overline{j}_2^t,\ldots,Y^t\}] \\
&\quad \times [x^*(t|t) \\
&\qquad - E\{x(t)|I^*(t-t_0),\gamma(t-t_0),\overline{\tau}_1^t,\overline{j}_1^t,\overline{\tau}_2^t,\overline{j}_2^t,\ldots,Y^t\}]'\,|I^*(t-t_0),Y^t\Big\} \\
&\cong \sum_{i_0=1}^{M}\Big\{\hat{P}(t|\gamma(t-t_0)=i_0,\overline{\tau}_1^t\geqq t) + [x^*(t|t) - \hat{x}(t|\gamma(t-t_0)=i_0,\overline{\tau}_1^t\geqq t)] \\
&\quad \times [x^*(t|t) - \hat{x}(t|\gamma(t-t_0)=i_0,\overline{\tau}_1^t\geqq t)]'\Big\}P(\gamma(t-t_0)=i_0,\overline{\tau}_1^t\geqq t|\overline{B}_{N_0}^t,Y^t) \\
&\quad + \sum_{n=1}^{N_0}\sum_{i_0=1}^{M}\cdots\sum_{\substack{i_n=1\\ i_n\neq i_{n-1}}}^{M}\int_{t-t_0}^{t}\int_{t_1}^{t}\cdots\int_{t_{n-1}}^{t}\Big\{\hat{P}(t|\gamma(t-t_0)=i_0,\overline{\tau}_t^n=t^n,\overline{j}_t^n=i^n,\overline{\tau}_{n+1}^t\geqq t) \\
&\quad + [x^*(t|t) - \hat{x}(t|\gamma(t-t_0)=i_0,\overline{\tau}_t^n=t^n,\overline{j}_t^n=i^n,\overline{\tau}_{n+1}^t\geqq t)] \\
&\quad \times [x^*(t|t) - \hat{x}(t|\gamma(t-t_0)=i_0,\overline{\tau}_t^n=t^n,\overline{j}_t^n=i^n,\overline{\tau}_{n+1}^t\geqq t)]'\Big\} \\
&\quad \times P(\gamma(t-t_0)=i_0,\overline{\tau}_t^n\varepsilon dt^n,\overline{j}_t^n=i^n,\overline{\tau}_{n+1}^t\geqq t|\overline{B}_{N_0}^t,Y^t),
\end{aligned} \tag{4.9}
$$

where

$$
\hat{P}(t|\gamma(t-t_0)=i_0,\overline{\tau}_1^t\geqq t) = \mathrm{Cov}\{x(t)|\gamma(t-t_0)=i_0,\overline{\tau}_1^t\geqq t,I^*(t-t_0),Y^t\} \tag{4.10$_1$}
$$

$$\hat{P}(t|\gamma(t-t_0)=i_0,\bar{\tau}_t^n=t^n,\bar{j}_t^n=i^n,\bar{\tau}_{n+1}^t\geq t)$$
$$= \mathrm{Cov}\{x(t)|\gamma(t-t_0)=i_0,\bar{\tau}_t^n=t^n,\bar{j}_t^n=i^n,\bar{\tau}_{n+1}^t\geq t,I^*(t-t_0),Y^t\}. \tag{4.10$_2$}$$

Moreover, by assumption (4.6), holds Lemma 2 where 0, τ_{n+1}, τ^n, $\gamma(0)$ and B_N^t are replaced by $t - t_0$, $\bar{\tau}_{n+1}^t$, $\bar{\tau}_t^n$, $\gamma(t-t_0)$ and $\bar{B}_{N_0}^t$, respectively.

For $t \leq t_0$, we can employ the approximate estimator algorithm presented in the previous subsection 3.4.1 to obtain the estimate $x^*(t|t)$ $(t \leq t_0)$, and the associated covariance matrix $P^*(t|t)$ is approximately given by

$$P^*(t|t) = \sum_{i_0=1}^{M}\Big\{\hat{P}(t|\gamma(0)=i_0,\tau_1\geq t) + [x^*(t|t) - \hat{x}(t|\gamma(0)=i_0,\tau_1\geq t)]$$
$$\times [x^*(t|t) - \hat{x}(t|\gamma(0)=i_0,\tau_1\geq t)]'\Big\}P(\gamma(0)=i_0,\tau_1\geq t|B_{N_0}^t,Y^t)$$
$$+ \sum_{n=1}^{N_0}\sum_{i_0=1}^{M}\cdots\sum_{\substack{i_n=1\\ i_n\neq i_{n-1}}}^{M}\int_0^t\int_{t_1}^t\cdots\int_{t_{n-1}}^t\Big\{\hat{P}(t|\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}\geq t)$$
$$+ [x^*(t|t) - \hat{x}(t|\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}\geq t)]$$
$$\times [x^*(t|t) - \hat{x}(t|\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}\geq t)]'\Big\}$$
$$\times P(\gamma(0)=i_0,\tau^n\varepsilon dt^n,j^n=i^n,\tau_{n+1}\geq t|B_{N_0}^t,Y^t) \tag{4.11}$$

where

$$\hat{P}(t|\gamma(0)=i_0,\tau_1\geq t) = \mathrm{Cov}\{x(t)|\gamma(0)=i_0,\tau_1\geq t,Y^t\} \tag{4.12$_1$}$$

$$\hat{P}(t|\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}\geqq t)$$
$$= \mathrm{Cov}\{x(t)|\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}\geqq t,Y^t\}. \qquad (4.12)_2$$

The values of $x^*(t|t)$ and $P^*(t|t)$ given by (4.3) and (4.11) for $t \leqq t_0$ and (4.7) and (4.9) for $t > t_0$ will be used as the a priori information to obtain the estimate $x^*(t+t_0|t+t_0)$ and the covariance matrix $P^*(t+t_0|t+t_0)$. For the implementation of the present algorithm, we also need the a posteriori probabilities $P(\gamma(t)=\ell|Y^t)$ (ℓ = 1, 2,..., M), which are approximately given, for $t \leqq t_0$, by

$$P(\gamma(t)=\ell|Y^t) \stackrel{\sim}{=} P(\gamma(t)=\ell|B^t_{N_0},Y^t)$$
$$= P(\gamma(0)=\ell,\tau_1\geqq t|B^t_{N_0},Y^t)$$
$$+ \sum_{n=1}^{N_0}\sum_{i_0=1}^{M}\cdots\sum_{\substack{i_{n-1}=1\\ i_{n-1}\neq\{i_{n-2},\ell\}}}^{M}\int_0^t\int_{t_1}^t\cdots\int_{t_{n-1}}^t P(\gamma(0)=i_0,\tau^n\varepsilon dt^n,j^{n-1}=i^{n-1},j_n=\ell|B^t_{N_0},Y^t) \qquad (4.13)$$

and for $t > t_0$, by

$$P(\gamma(t)=\ell|Y^t)$$
$$\stackrel{\sim}{=} P(\gamma(t-t_0)=\ell,\bar{\tau}^t_1\geqq t|\bar{B}^t_{N_0},Y^t)$$
$$+ \sum_{n=1}^{N_0}\sum_{i_0=1}^{M}\cdots\sum_{\substack{i_{n-1}=1\\ i_{n-1}\neq\{i_{n-2},\ell\}}}^{M}\int_{t-t_0}^t\int_{t_1}^t\cdots$$

$$\cdots \int_{t_{n-1}}^{t} P(\gamma(t-t_0)=i_0, \bar{\tau}_t^n \varepsilon dt^n, \bar{j}_t^{n-1}=i^{n-1}, \bar{j}_n^t=\ell \mid \bar{B}_{N_0}^t, Y^t). \quad (4.14)$$

Equations (4.3) and (4.11)-(4.13) for $t \leqq t_0$ and (4.7)-(4.10) and (4.14) for $t > t_0$ together with Lemma 2 completely specify the approximate estimator algorithm for the case of the infinite interval of operation.

Remark 1 : It should be noted that the approximate estimator algorithm presented here is based upon assumption (4.6) and that there is a possibility of divergence of the estimation error. It should also be noted that from a theoretical point of view if $N_0 \to \infty$ and $\varepsilon \to +0$ simultaneously, then the approximate estimator algorithm presented here converges to the optimal one over the every finite time interval of operation with probability 1.

Remark 2 : If, for a given small number ε, the integer N_0 defined by (4.1) is too large for the available computer memory, then the approximate estimator algorithm for the infinite interval of operation is recommendable even for the case of the finite interval of operation.

3.5 Special Case

In this section, we shall make a specialization of the results obtained in the previous sections for the following systems: (i) linear continuous systems with system-component failure, and (ii) linear continuous systems with interrupted observations.

3.5.1 Linear Continuous System with System-Component Failure

Let us consider the state estimation problem for systems with random component failure; that is, the system is represented by a stochastic differential equation:

$$dx(t) = F(t,\gamma(t))x(t)dt + G(t,\gamma(t))u(t)dt + Q(t,\gamma(t))dw(t), \tag{5.1}$$

and the observation is given by

$$dy(t) = H(t)x(t)dt + R(t)dv(t). \tag{5.2}$$

Here, the difference between the present system (5.1) and (5.2) and the original system (2.1) and (2.2) is that the jump process $\gamma(t)$ does not appear in the observation equation (5.2).

For system (5.1) and (5.2), the minimum-variance estimate

$\hat{x}(t|t)$ is given by the following.

Theorem 3.2

For the system described by (5.1) and (5.2), the minimum-variance estimate $\hat{x}(t|t)$ of state $x(t)$ given observations Y^t is furnished by

$$\hat{x}(t|t) = \sum_{i_0=1}^{M} \hat{x}(t|\gamma(0)=i_0,\tau_1 \geq t)P(\gamma(0)=i_0,\tau_1 \geq t|Y^t)$$

$$+ \sum_{n=1}^{\infty} \sum_{i_0=1}^{M} \cdots \sum_{\substack{i_n=1 \\ i_n \neq i_{n-1}}}^{M} \int_0^t \int_{t_1}^t \cdots \int_{t_{n-1}}^t \hat{x}(t|\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1} \geq t)$$

$$\times P(\gamma(0)=i_0,\tau^n \epsilon dt^n,j^n=i^n,\tau_{n+1} \geq t|Y^t), \quad (5.3)$$

where $\hat{x}(t|\gamma(0)=i_0,\tau_1 \geq t)$ and $\hat{x}(t|\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1} \geq t|Y^t)$ are defined by (3.2) and are given by the usual Kalman-filter algorithms. The a posteriori probabilities $P(\gamma(0)=i_0,\tau_1 \geq t|Y^t)$ and $P(\gamma(0)=i_0,\tau^n \epsilon dt^n,j^n=i^n,\tau_{n+1} \geq t|Y^t)$ are given by

$$P(\gamma(0)=i_0,\tau_1 \geq t|Y^t) = \frac{L(t,\{\gamma(0)=i_0,\tau_1 \geq t\})P(\gamma(0)=i_0,\tau_1 \geq t)}{L(t)} \quad (5.4)$$

and

$$P(\gamma(0)=i_0,\tau^n \epsilon dt^n,j^n=i^n,\tau_{n+1} \geq t|Y^t)$$

$$= \frac{L(t,\{\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1} \geq t\})P(\gamma(0)=i_0,\tau^n \epsilon dt^n,j^n=i^n,\tau_{n+1} \geq t)}{L(t)} \quad (5.5)$$

where

$$L(t) = \exp\left\{\int_0^t \hat{x}'(s|s)H'(s)\{R(s)R'(s)\}^{-1}dy(s) - \frac{1}{2}\int_0^t \|H(s)\hat{x}(s|s)\|^2_{\{R(s)R'(s)\}^{-1}} ds\right\} \tag{5.6}$$

$$L(t,\{\gamma(s),0\leqq s<t\}) = \exp\left\{\int_0^t \hat{x}'(s|\gamma(u),0\leqq u<s)H'(s)\{R(s)R'(s)\}^{-1}dy(s) - \frac{1}{2}\int_0^t \|H(s)\hat{x}(s|\gamma(u),0\leqq u<s)\|^2_{\{R(s)R'(s)\}^{-1}} ds\right\} \tag{5.7}$$

and

$$\hat{x}(s|\gamma(u),0\leqq u<s) \triangleq E\{x(s)|\gamma(u),0\leqq u<s,Y^s\}.$$

Here, the likelihood ratios $L(t,\{\gamma(0)=i_0,\tau_1\geqq t\})$ and $L(t,\{\gamma(0)=i_0,\tau^n=t^n,j^n=i^n,\tau_{n+1}\geqq t\})$ are obtained by substituting the respective values of $\gamma(s)$ $(0\leqq s<t)$ specified in the brackets into (5.7). Moreover, the a priori probabilities $P(\gamma(0)=i_0,\tau_1\geqq t)$ and $P(\gamma(0)=i_0,\tau^n\varepsilon dt^n,j^n=i^n,\tau_{n+1}\geqq t)$ appearing in (5.4) and (5.5) are given by (3.7) and (3.8).

Remark : For the present case, the expression (5.6) of the likelihood ratio $L(t)$ is significantly simplified, compared with equation (3.6) of the original system. This is due to the fact that the jump process $\gamma(t)$ does not appear in the observation equation (5.2).

3.5.2 Linear Continuous System with Interrupted Observation

We shall now consider the estimation problem for linear continuous systems with the interrupted observation mechanism which is characterized in terms of the jump Markov process taking on values of 0 or 1. Consider the system represented by a stochastic differential equation:

$$dx(t) = F(t)x(t)dt + Q(t)dw(t), \tag{5.8}$$

and let the interrupted observation be given by

$$dy(t) = \gamma(t)H(t)x(t)dt + R(t)dv(t). \tag{5.9}$$

The present system is the continuous-time counterpart of the discrete system treated in section 2.7, chapter II. Note that system equation (5.8) does not contain the jump process $\gamma(t)$ and that the jump process $\gamma(t)$ takes on values of 0 or 1 only. From (5.9), if $\gamma(t)$ is equal to 1, the observation process contains the information about the state $x(t)$; while if $\gamma(t)$ is equal to 0, then the observation process consists of noise only. Thus the jump process $\gamma(t)$ characterizes the interrupted observation mechanisms and is called interruption process as in the case of discrete systems treated in section 2.7, chapter II.

The minimum-variance estimator algorithm for system (5.8) and (5.9) is given by the following.

Theorem 3.3

The minimum-variance estimate $\hat{x}(t|t)$ for system (5.8) and (5.9) is given by

$$\hat{x}(t|t) = \sum_{i=0}^{1}\Big\{\hat{x}_i(t|t)P(\tau_1 \geqq t|\gamma(0)=i,Y^t)P(\gamma(0)=i|Y^t)$$

$$+ \sum_{n=1}^{\infty}\int_0^t\int_{t_1}^t\cdots\int_{t_{n-1}}^t \hat{x}_i(t|t,t^n)p(\tau^n=t^n|\gamma(0)=i,Y^t)$$

$$\times P(\tau_{n+1} \geqq t|\gamma(0)=i,\tau^n=t^n,Y^t)P(\gamma(0)=i|Y^t)dt_n\cdots dt_1\Big\}, \tag{5.10}$$

where $\hat{x}_i(t|t)$ and $\hat{x}_i(t|t,t^n)$ are defined by

$$\hat{x}_i(t|t) = E\{x(t)|\gamma(0)=i,\tau_1 \geqq t,Y^t\} \qquad (5.11)_1$$

and

$$\hat{x}_i(t|t,t^n) = E\{x(t)|\gamma(0)=i,\tau^n=t^n,\tau_{n+1} \geqq t\}. \qquad (5.11)_2$$

Also, $P(\tau_1 \geqq t|\gamma(0)=i,Y^t)P(\gamma(0)=i|Y^t)$ and $p(\tau^n=t^n|\gamma(0)=i,Y^t)$ $\cdot P(\tau_{n+1} \geqq t|\gamma(0)=i,\tau^n=t^n,Y^t)P(\gamma(0)=i|Y^t)$ are given by

$$P(\tau_1 \geqq t|\gamma(0)=i,Y^t)P(\gamma(0)=i|Y^t)$$

$$= \frac{L(t,\{\gamma(0)=i,\tau_1 \geqq t\})P(\tau_1 \geqq t|\gamma(0)=i)P(\gamma(0)=i)}{L(t)} \tag{5.12}$$

and

$$p(\tau^n=t^n|\gamma(0)=i,Y^t)P(\tau_{n+1}\geq t|\gamma(0)=i,\tau^n=t^n,Y^t)P(\gamma(0)=i|Y^t)$$

$$= \frac{L(t,\{\gamma(0)=i,\tau^n=t^n,\tau_{n+1}\geq t\})p(\tau^n=t^n|\gamma(0)=i)P(\tau_{n+1}\geq t|\gamma(0)=i,\tau_n=t_n)}{L(t)} * $$

$$* \frac{P(\gamma(0)=i)}{\quad} \tag{5.13}$$

where

$$L(t,\{\gamma(0)=i,\tau_1\geq t\})$$

$$= \exp\left\{\int_0^t i\cdot\hat{x}_i'(s|s)H'(s)\{R(s)R'(s)\}^{-1}dy(s) - \frac{1}{2}\int_0^t i\cdot\|H(s)\hat{x}_i(s|s)\|^2_{\{R(s)R'(s)\}^{-1}}\,ds\right\}, \tag{5.14}$$

$$L(t,\{\gamma(0)=i,\tau^n=t^n,\tau_{n+1}\geq t\})$$

$$= \exp\left\{\int_0^t \gamma_i(s,t^n)\hat{x}_i'(s|s,t^n)H'(s)\{R(s)R'(s)\}^{-1}dy(s) - \frac{1}{2}\int_0^t \gamma_i(s,t^n)\|H(s)\hat{x}_i(s|s,t^n)\|^2_{\{R(s)R'(s)\}^{-1}}\,ds\right\} \tag{5.15}$$

and

$$L(t) = \sum_{i=0}^{1}\Big\{L(t,\{\gamma(0)=i,\tau_1\geq t\})P(\tau_1\geq t|\gamma(0)=i)P(\gamma(0)=i) +$$

$$+ \sum_{n=1}^{\infty}\int_0^t\int_{t_1}^t\cdots\int_{t_{n-1}}^t L(t,\{\gamma(0)=i,\tau^n=t^n,\tau_{n+1}\geqq t\})p(\tau^n=t^n|\gamma(0)=i)$$

$$\times P(\tau_{n+1}\geqq t|\gamma(0)=i,\tau_n=t_n)P(\gamma(0)=i)dt_n\cdots dt_1\Big\}. \quad (5.16)$$

Here, $\gamma_i(s,t^n)$ appearing in (5.15) is $\gamma(s)$ with $\gamma(0) = i$, $\tau^n=t^n$, $\tau_{n+1}\geqq t$ and $0 < t_1 < \cdots < t_n < t$. Moreover,

$$P(\tau_1\geqq t|\gamma(0)=i) = \exp\left\{ - \int_0^t q_{ij}(s)ds\right\}, \quad (5.17)$$

$$P(\tau_{n+1}\geqq t|\gamma(0)=i,\tau_n=t_n) = \exp\left\{ - \int_{t_n}^t q_{ij}(s)ds\right\}, \text{ for } n = 2m,$$

$$= \exp\left\{ - \int_{t_n}^t q_{ji}(s)ds\right\}, \text{ for } n = 2m-1, \quad (5.18)$$

and

$$p(\tau^n=t^n|\gamma(0)=i)$$

$$= \exp\left\{-\int_0^{t_1} q_{ij}(s)ds - \int_{t_1}^{t_2} q_{ji}(s)ds - \cdots - \int_{t_{n-1}}^{t_n} q_{ji}(s)ds\right\}$$

$$\times q_{ij}(t_1)q_{ji}(t_2)\cdots q_{ji}(t_n), \text{ for } n = 2m,$$

$$= \exp\left\{-\int_0^{t_1} q_{ij}(s)ds - \int_{t_1}^{t_2} q_{ji}(s)ds - \cdots - \int_{t_{n-1}}^{t_n} q_{ij}(s)ds\right\}$$

$$\times q_{ij}(t_1)q_{ji}(t_2)\cdots q_{ij}(t_n), \text{ for } n = 2m-1, \quad (5.19)$$

where $i, j = 0, 1, \ i \neq j$, and $m = 1, 2, \ldots$.

Remark : For the present case, the interruption process $\gamma(t)$ takes on values of 0 or 1 only, so that we can express the interruption process $\gamma(t)$ in terms of its initial value $\gamma(0)$ and the jump times τ_n without using the random variable j_n defined by (2.5). Due to this fact, the optimal estimator algorithm for the present case is slightly simplified, compared with the general one presented in section 3.3.

3.6 Numerical Example

We shall carry out simulation studies in order to illustrate the behavior of the optimal estimator and to demonstrate the feasibility of the proposed approximate estimator algorithms.

Example 1

Consider the scalar system:

$$dx(t) = f(\gamma(t))x(t)dt + g(\gamma(t))u(t)dt + Q(t)dw(t) \tag{6.1}$$

$$dy(t) = x(t)dt + R(t)dv(t). \tag{6.2}$$

Here, it is assumed that the jump Markov process $\gamma(t)$ is stationary, that is, $q_{i_{k-1},i_k}(t) \equiv q_{i_{k-1},i_k}$ and takes on values of 1, 2, 3 and that $q_{2,1} = 0$, $q_{3,1} = 0$, $q_{2,3} = 0$, $q_{3,2} = 0$. This means that only the jumps from 1 to 2 and from 1 to 3 are possible. Also, we assume $\gamma(0) = 1$ with probability 1. In this case, for example, if $\gamma(t) = 1$, the system is in normal operation and if $\gamma(t) = 2$ or 3, the failure of machine A or machine B has occured, respectively. The system described by (6.1) and (6.2) is the one treated in section 3.5.1.

Since the jump may occur at most once in this example, from (5.3) the optimal estimate $\hat{x}(t|t)$ is given by

$$\begin{aligned}\hat{x}(t|t) &= \hat{x}(t|\gamma(0)=1,\tau_1 \geqq t)P(\tau_1 \geqq t|\gamma(0)=1,Y^t) \\ &\quad + \sum_{i_1=2}^{3}\int_0^t \hat{x}(t|\gamma(0)=1,\tau_1=t_1,j_1=i_1) \\ &\qquad\qquad \times P(\tau_1 \varepsilon dt_1, j_1=i_1|\gamma(0)=1,Y^t), \qquad (6.3)\end{aligned}$$

where

$$P(\tau_1 \geqq t|\gamma(0)=1,Y^t) = \frac{L(t,\{\gamma(0)=1,\tau_1 \geqq t\})P(\tau_1 \geqq t|\gamma(0)=1)}{L(t)} \qquad (6.4)$$

$$\begin{aligned}&P(\tau_1 \varepsilon dt_1, j_1=i_1|\gamma(0)=1,Y^t) \\ &= \frac{L(t,\{\gamma(0)=1,\tau_1=t_1,j_1=i_1\})P(\tau_1 \varepsilon dt_1,j_1=i_1|\gamma(0)=1)}{L(t)}. \qquad (6.5)\end{aligned}$$

Here, $L(t,\{\gamma(0)=1,\tau_1 \geqq t\})$ and $L(t,\{\gamma(0)=1,\tau_1=t_1,j_1=i_1\})$ are given by the formula (5.7) and $L(t)$ by (5.6).

Computer simulation is carried out over the time interval [0,1] using the following numerical values:

$$f(1) = -4.5, \quad f(2) = 0, \quad f(3) = 5.0, \quad g(1) = 1.0,$$
$$g(2) = 0, \quad g(3) = 0, \quad Q^2(t) = 0.25, \quad R^2(t) = 0.25,$$
$$q_{1,2} = 1.0, \quad q_{1,3} = 1.0, \quad \hat{x}(0|0) = 5.0, \quad \hat{P}(0|0) = 10,$$
$$u(t) = -1.0, \quad x(0)=10,$$

and we set the time increment $\Delta t = 0.01$.

Fig. 3.1 shows the sample behavior of the state $x(t)$ and the optimal estimate $\hat{x}(t|t)$. The corresponding behavior of the jump process $\gamma(t)$ and the a posteriori probabilities $P(\gamma(t)=1|Y^t)$, $P(\gamma(t)=2|Y^t)$ and $P(\gamma(t)=3|Y^t)$ are shown in Fig. 3.2, where the a posteriori probabilities are given by

$$P(\gamma(t)=1|Y^t) = P(\tau_1 \geqq t|\gamma(0)=1,Y^t) \tag{6.6}$$

and

$$P(\gamma(t)=i_1|Y^t) = \int_0^t P(\tau_1 \varepsilon dt_1, j_1=i_1|\gamma(0)=1,Y^t), \quad i_1 = 2,\ 3 \tag{6.7}$$

together with (6.4) and (6.5). It should be noted that we can perform filtering and detection of the jump process $\{\gamma(t), 0\leqq t<1\}$

by using (6.6) and (6.7).

Example 2

Let us now consider the following scalar system with the interrupted observation mechanism:

$$dx(t) = -ax(t)dt + Q(t)dw(t) \tag{6.8}$$

$$dy(t) = \gamma(t)x(t)dt + R(t)dv(t), \tag{6.9}$$

where $a > 0$, and let $q_{1,0}(t) \equiv q_{1,0}$, $q_{0,1}(t) \equiv q_{0,1}$ and $P(\gamma(0)=1) = p_1$. System (6.8) and (6.9) corresponds to the system treated in section 3.5.2.

In the case when $p_1 = 1$ and $q_{0,1} = 0$, from (5.10) the optimal estimate $\hat{x}(t|t)$ is given by

$$\begin{aligned}\hat{x}(t|t) &= \hat{x}_1(t|t)P(\tau_1 \geq t|\gamma(0)=1,Y^t) \\ &\quad + \int_0^t \hat{x}_1(t|t,t_1)p(\tau_1=t_1|\gamma(0)=1,Y^t)dt_1.\end{aligned} \tag{6.10}$$

Computer simulation is carried out over the time interval [0,1] using the following set of numerical values:

$$\begin{aligned}&a = 1.5, \quad Q^2(t) = 1.0, \quad R^2(t) = 0.1, \quad q_{1,0} = 2.0, \\ &q_{0,1} = 0, \quad x(0) = 10, \quad \hat{x}(0|0) = 5, \quad \hat{P}(0|0) = 10, \\ &p_1 = 1.0,\end{aligned} \tag{6.11}$$

and we set the time increment $\Delta t = 0.01$.

Fig. 3.3 shows the tracking behavior of the state $x(t)$ and the optimal estimate $\hat{x}(t|t)$ together with the a posteriori probability $P(\gamma(t)=1|Y^t)$, where

$$P(\gamma(t)=1|Y^t) = P(\tau_1 \geq t|\gamma(0)=1,Y^t)$$

with probability 1.

Next, we consider the case when $q_{1,0} \neq 0$ and $q_{0,1} \neq 0$. In this case, the optimal estimator algorithm is not feasible, because the number of jumps which occur in the time interval [0,1] is not bounded with probability 1. The approximate estimate $x^*(t|t)$ presented in section 3.4.2 is given by

$$x^*(t|t) = \sum_{i=0}^{1}\Big\{\hat{x}_i(t|t)P(\tau_1 \geq t|\gamma(0)=i,B_1^t,Y^t)P(\gamma(0)=i|B_1^t,Y^t)$$

$$+ \int_0^t \hat{x}_i(t|t,t_1)p(\tau_1=t_1|\gamma(0)=i,B_1^t,Y^t)P(\gamma(0)=i|B_1^t,Y^t)dt_1\Big\}$$

for $t \leq t_0$, and

$$x^*(t|t) = \sum_{i=0}^{1}\Big\{\hat{x}_i(t|t,\bar{\tau}_t^0)P(\bar{\tau}_1^t \geq t|\gamma(t-t_0)=i,\bar{B}_1^t,Y^t)P(\gamma(t-t_0)=i|\bar{B}_1^t,Y^t)$$

$$+ \int_{t-t_0}^{t} \hat{x}_i(t|t,\bar{\tau}_1^t=t_1)p(\bar{\tau}_1^t=t_1|\gamma(t-t_0)=i,\bar{B}_1^t,Y^t)P(\gamma(t-t_0)=i|\bar{B}_1^t,Y^t)dt_1\Big\}$$

for $t > t_0$, where

$$\hat{x}_i(t|t,\bar{\tau}_t^0) \triangleq E\{x(t)|\gamma(t-t_0)=i,\bar{\tau}_1^t \geq t, Y^t\},$$

$$\hat{x}_i(t|t,\bar{\tau}_1^t=t_1) \triangleq E\{x(t)|\gamma(t-t_0)=i,\bar{\tau}_1^t=t_1,\bar{\tau}_2^t \geq t, Y^t\}$$

and the integer N_0 appearing in (4.4) is chosen to be 1.

Computer simulation is also carried out using the set of numerical values in (6.11) except that $q_{0,1} = 2.0$ and $p_1 = 0.95$. We set $\varepsilon = 0.018$ and $t_0 = 0.1$ so that inequality (4.4) is satisfied. Fig. 3.4 shows the tracking behavior of the state $x(t)$, the approximate estimate $x^*(t|t)$ and the estimate $\bar{x}(t|t)$ using the Kalman-filter algorithm when the true values of $\gamma(t)$ are known, together with the approximate a posteriori probability of $P(\gamma(t)=1|Y^t)$. We may observe from Fig. 3.4 that the approximate estimates $x^*(t|t)$ are nearly optimal even when the integer N_0 is chosen to be 1.

3.7 Concluding Remarks

The minimum-variance estimator algorithm is presented for linear continuous systems with a multi-valued jump Markov process. In the case when the jump Markov process has an ergodic subclass, the optimal estimator algorithm is not feasible because the summation of infinitely many terms must be carried out. This is due to the fact that the number of possible jumps is not bounded. Therefore, the feasible approximate estimator algorithms are proposed for the practical implementation. It should, however, be noted that, to implement the approximate estimator algorithms, time is quantized to a level determined by the characteristics of the jump Markov process under consideration and the random jump is assumed to occur at the quantized time. It should also be noted that the quantization of the jump time is performed in the approximate estimator algorithms and that if from the beginning the random jump is assumed to occur at the quantized time in original system (2.1) and (2.2), then the estimation problem considered here becomes similar to that of multi-shot joint detection and estimation considered by Lainiotis (1972). Finally, it should be noted that we can also perform filtering, smoothing and detection of the jump process $\{\gamma(t), 0 \leq t\}$ by using the a posteriori probability distribution (3.3) and (3.4) of the jump process.

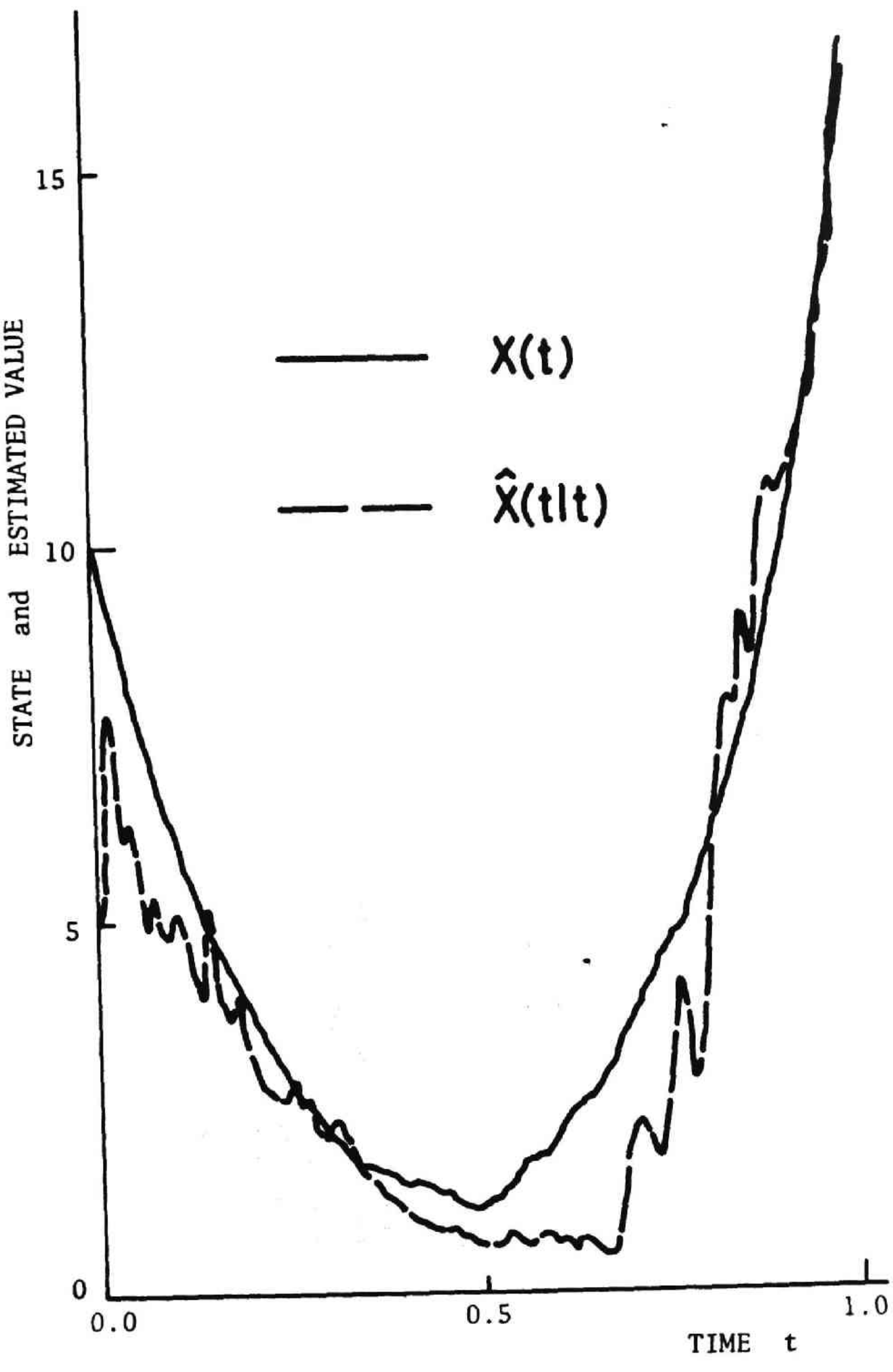

Fig. 3.1. Sample paths of $\hat{x}(t|t)$ and $x(t)$.

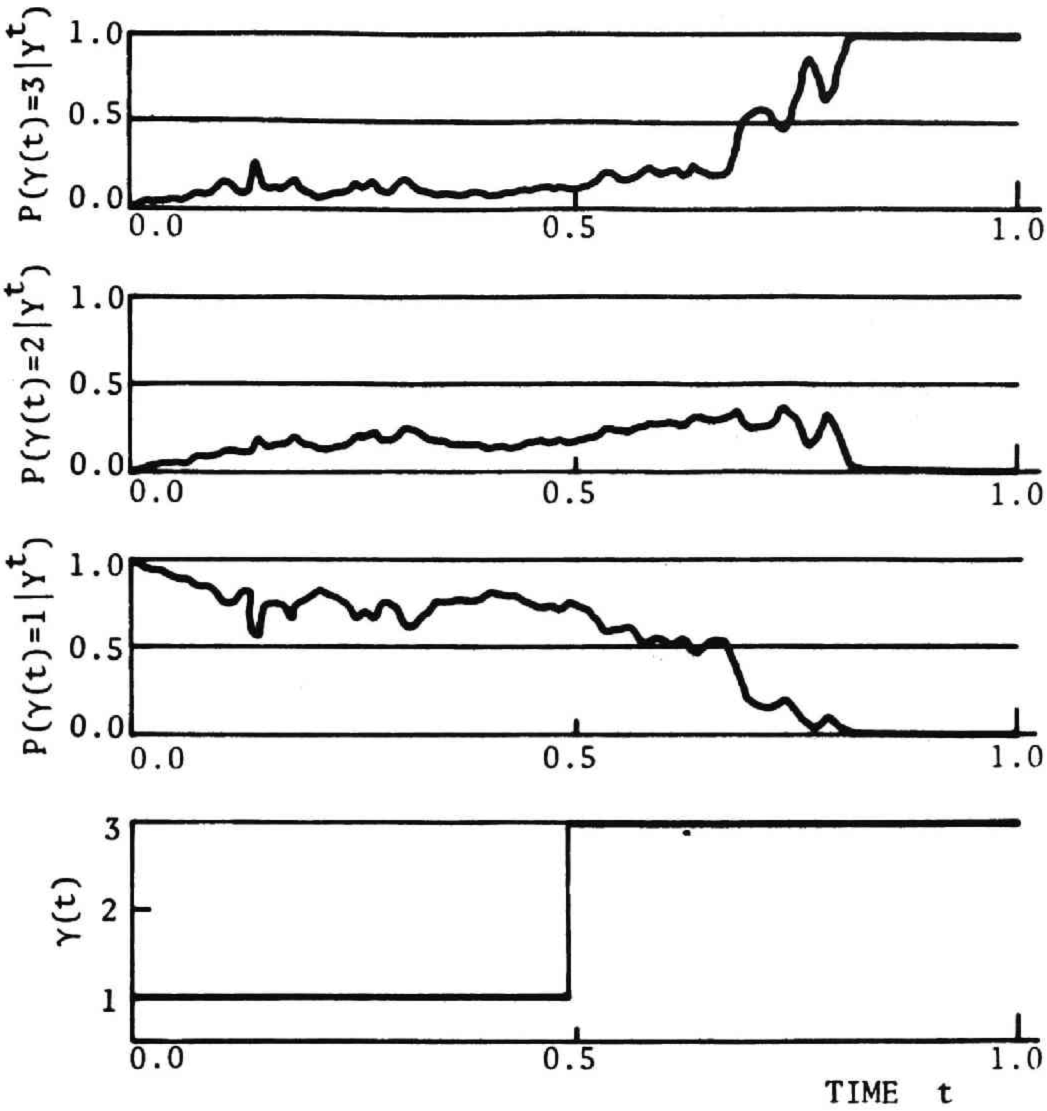

Fig. 3.2. A sample path of $\gamma(t)$ and the corresponding behavior of $P(\gamma(t)=i|Y^t)$ $(i=1,2,3)$.

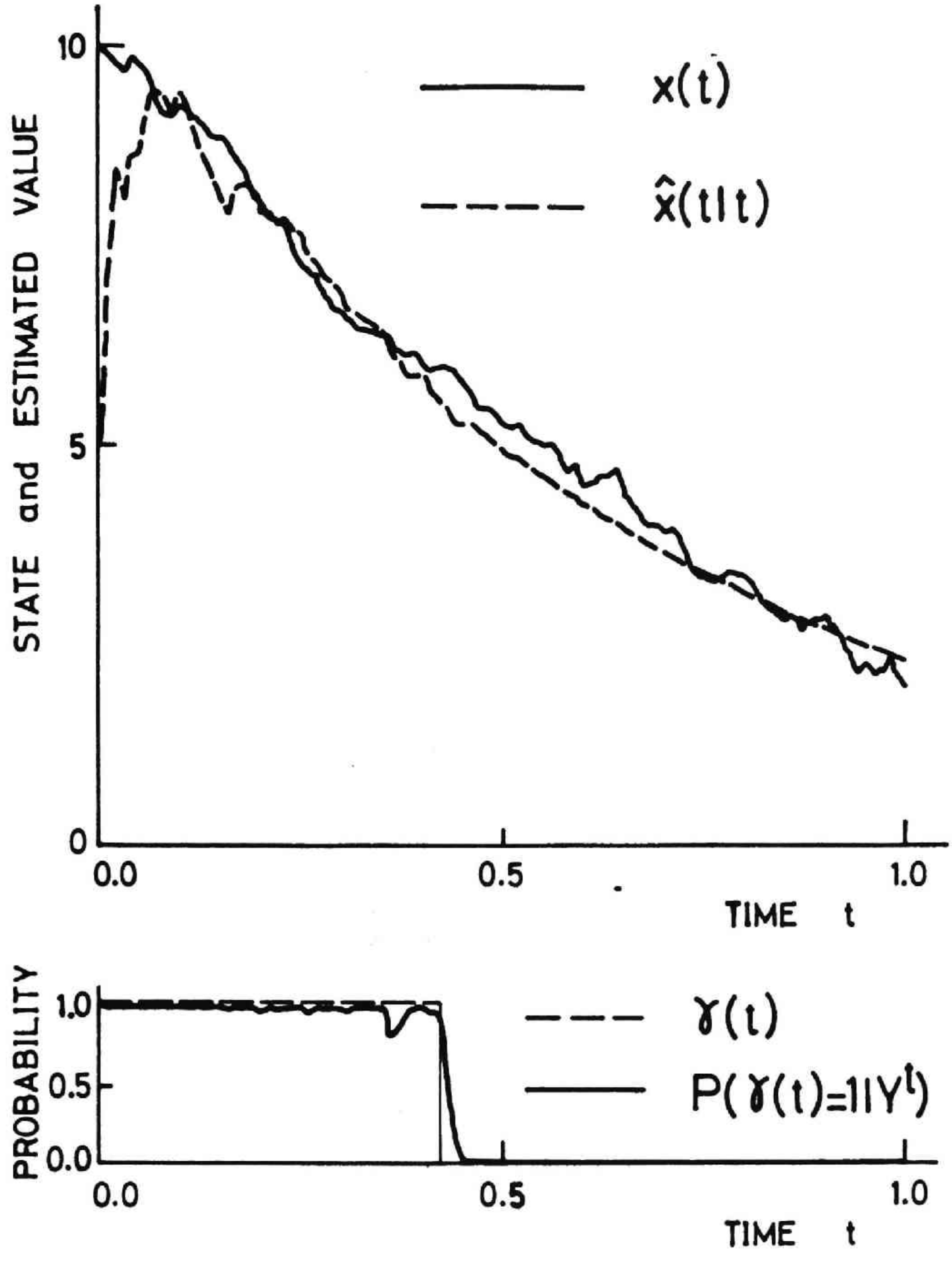

Fig. 3.3. Sample paths of $\hat{x}(t|t)$ and $x(t)$, and the a posteriori probability $P(\gamma(t)=1|Y^t)$.

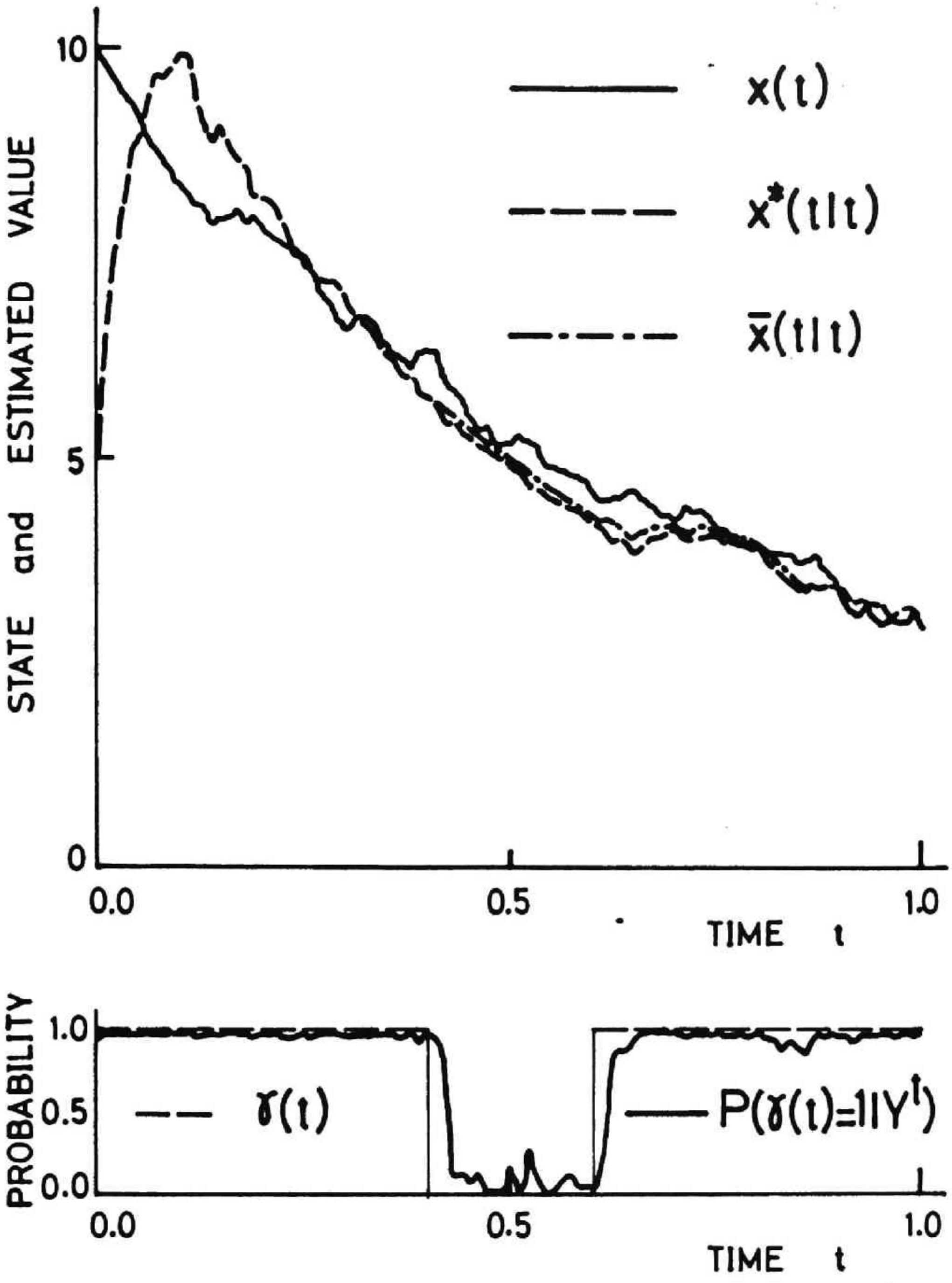

Fig. 3.4. Sample paths of $x^*(t|t)$, $\overline{x}(t|t)$ and $x(t)$, and the approximate a posteriori probability $P(\gamma(t)=1|Y^t)$.

Appendix I

Proof of Lemma 1

Let $\theta \triangleq (\theta_0, \theta_1, \ldots, \theta_N)$ and $\alpha \triangleq (\alpha_1, \alpha_2, \ldots, \alpha_N)$ be random variables defined by

$$\left.\begin{aligned} \theta &= (i, \tau^*, \tau^*, \ldots, \tau^*) \\ \alpha &= (\tau^*, \tau^*, \ldots, \tau^*) \end{aligned}\right\} \quad \text{if } \gamma(0) = i, \quad \tau_1 \geqq t ,$$

$$\left.\begin{aligned} \theta &= (i, t_1, t_2, \ldots, t_n, \tau^*, \tau^*, \ldots, \tau^*) \\ \alpha &= (i_1, i_2, \ldots, i_n, \tau^*, \tau^*, \ldots, \tau^*) \end{aligned}\right\} \quad \begin{aligned} &\text{if } \gamma(0) = i, \quad \tau^n = t^n, \\ &\quad j^n = i^n, \quad \tau_{n+1} \geqq t, \end{aligned}$$

and

$$\left.\begin{aligned} \theta &= (0, 0, \ldots, 0) \\ \alpha &= (0, 0, \ldots, 0) \end{aligned}\right\} \quad \text{if } \tau_{N+1} \geqq t, \qquad \text{(A.1)}$$

where $0 < t_1 < t_2 < \cdots < t_n < t$, $n = 1, 2, \ldots, N$ and τ^* is an arbitrary but fixed real value in $[t, \infty)$. From this definition, if θ $(\neq (0,0,\ldots,0))$ and α $(\neq (0,0,\ldots,0))$ are known, the behavior of $\{\gamma(s), 0 \leqq s < t\}$ is completely specified. Therefore, if we confine the elementary events ω in $B_N^t \subset \Omega$, that is, if we assume that the number of jumps of $\gamma(s)$ $(0 \leqq s < t)$ is less than N, then we can regard the random variables θ and α as the $N+1$- and N-dimensional unknown constant parameter vectors for linear system (2.1) and (2.2), respectively. Thus, from Lainiotis' formula (Lainiotis, 1971),

$$p(\theta,\alpha|B_N^t,Y^t)d\theta d\alpha = \frac{L(t,\{\theta,\alpha\})p(\theta,\alpha|B_N^t)d\theta d\alpha}{\int_{R^{2N+1}} L(t,\{\theta,\alpha\})p(\theta,\alpha|B_N^t)d\theta d\alpha} \tag{A.2}$$

where

$$L(t,\{\theta,\alpha\}) = \exp\left\{\int_0^t \hat{x}'(s|s,\{\theta,\alpha\})H'(s,\{\theta,\alpha\})\{R(s)R'(s)\}^{-1}dy(s) - \frac{1}{2}\int_0^t \|H(s,\{\theta,\alpha\})\hat{x}(s|s,\{\theta,\alpha\})\|^2_{\{R(s)R'(s)\}^{-1}}\,ds\right\} \tag{A.3}$$

and

$$\hat{x}(s|s,\{\theta,\alpha\}) \triangleq E\{x(s)|\theta,\alpha,Y^s\}.$$

Here, $H(s,\{\theta,\alpha\})$ is defined as $H(s,\gamma(s))$ corresponding to $\{\theta,\alpha\}$, and R^{2N+1} is the 2N+1-dimensional Euclidean space. Likelihood ratio (A.3) exists under the assumption (2.8). By the definition (A.1) of θ and α, the conditional joint probability density function $p(\theta,\alpha|B_N^t)$ of θ and α given B_N^t is expressed as

$$\begin{aligned} &p(\theta,\alpha|B_N^t)d\theta d\alpha \\ &= \sum_{i_0=1}^{M} P(\gamma(0)=i_0,\tau_1\geq t|B_N^t)\delta(\theta_0-i_0)\prod_{k=1}^{N}\{\delta(\theta_k-\tau^*)\delta(\alpha_k-\tau^*)\}d\theta_0 d\theta^N d\alpha^N \\ &+ \sum_{n=1}^{N-1}\sum_{i_0=1}^{M}\cdots\sum_{\substack{i_n=1 \\ i_n\neq i_{n-1}}}^{M} P(\gamma(0)=i_0,\tau^n\epsilon d\theta^n,j^n=i^n,\tau_{n+1}\geq t|B_N^t) \\ &\qquad\times\delta(\theta_0-i_0)\prod_{j=1}^{n}\delta(\alpha_j-i_j)\prod_{k=n+1}^{N}\{\delta(\theta_k-\tau^*)\delta(\alpha_k-\tau^*)\}\times \end{aligned}$$

$$\times \prod_{j=1}^{n} u_t(\theta_j)d\theta_0 d\theta_{n+1}\cdots d\theta_N d\alpha^N$$

$$+ \sum_{i_0=1}^{M} \cdots \sum_{\substack{i_n=1 \\ i_n \neq i_{n-1}}}^{M} P(\gamma(0)=i_0, \tau^N \epsilon d\theta^N, j^N = i^N | B_N^t)\delta(\theta_0 - i_0)$$

$$\times \prod_{j=1}^{N} \{\delta(\alpha_j - i_j)u_t(\theta_j)\}d\theta_0 d\alpha^N, \qquad \text{(A.4)}$$

where $\theta^n = (\theta_1, \theta_2, \ldots, \theta_n)$, $\alpha^n = (\alpha_1, \alpha_2, \ldots, \alpha_n)$,

$$u_t(s) = \begin{cases} 1 & \text{for} \quad 0 < s < t \\ 0 & \text{otherwise,} \end{cases} \qquad \text{(A.5)}$$

and $\delta(\cdot)$ denotes Dirac delta function.

Also, the a posteriori conditional probability density function $p(\theta, \alpha | B_N^t, Y^t)$ becomes

$$p(\theta, \alpha | B_N^t, Y^t)d\theta d\alpha$$

$$= \sum_{i_0=1}^{M} P(\gamma(0)=i_0, \tau_1 \geq t | B_N^t, Y^t)\delta(\theta_0 - i_0)$$

$$\times \prod_{k=1}^{N} \{\delta(\theta_k - \tau^*)\delta(\alpha_k - \tau^*)\}d\theta_0 d\theta^N d\alpha^N$$

$$+ \sum_{n=1}^{N-1} \sum_{i_0=1}^{M} \cdots \sum_{\substack{i_n=1 \\ i_n \neq i_{n-1}}}^{M} P(\gamma(0)=i_0, \tau^n \epsilon d\theta^n, j^n = i^n, \tau_{n+1} \geq t | B_N^t, Y^t)$$

$$\times \delta(\theta_0 - i_0) \prod_{j=1}^{n} \delta(\alpha_j - i_j) \prod_{k=n+1}^{N} \{\delta(\theta_k - \tau^*)\delta(\alpha_k - \tau^*)\} \times$$

$$\times \prod_{j=1}^{n} u_t(\theta_j)d\theta_0 d\theta_{n+1}\cdots d\theta_N d\alpha^N$$

$$+ \sum_{i_0=1}^{M} \cdots \sum_{\substack{i_n=1 \\ i_n \neq i_{n-1}}}^{M} P(\gamma(0)=i_0, \tau^N \varepsilon d\theta^N, j^N=i^N | B_N^t, Y^t)\, \delta(\theta_0 - i_0)$$

$$\times \prod_{j=1}^{N} \{\delta(\alpha_j - i_j) u_t(\theta_j) d\theta_0 d\alpha^N. \qquad (A.6)$$

Substituting (A.4) and (A.6) into (A.2) and comparing the both sides of the resultant equation yields Lemma 1, where $\gamma(0)$, τ^n and j^n are used instead of θ_0, θ^n and α^n, in view of definition (A.1).

Appendix II

Proof of Lemma 2

By the definition of conditional probability,

$$P(\gamma(0)=i_0, \tau^n \varepsilon dt^n, j^n=i^n, \tau_{n+1} \geq t | B_N^t)$$

$$= \frac{P(\gamma(0)=i_0, \tau^n \varepsilon dt^n, j^n=i^n, \tau_{n+1} \geq t)}{P(B_N^t)} \qquad (A.7)$$

holds for $n = 1, 2, \ldots, N$, and similarly,

$$P(\gamma(0)=i_0, \tau_1 \geq t | B_N^t) = \frac{P(\gamma(0)=i_0, \tau_1 \geq t)}{P(B_N^t)}. \qquad (A.8)$$

Substituting (A.8) and (A.7) into (3.21) and (3.22), respectively, yields Lemma 2.

CHAPTER IV

STATE ESTIMATION FOR LINEAR CONTINUOUS-DISCRETE SYSTEM WITH JUMP PROCESS

4.1 Introduction

In practical situations, especially in computer control systems, the observations are usually given at discrete-time instants; while most physical processes have dynamical behavior best described using differential equations. Thus, in this chapter, we consider the state estimation problem for a class of linear continuous-discrete systems with a jump process, where the system is described by a stochastic differential equation and the observations are made at discrete times.

The minimum-variance estimator algorithm will be derived here. The approach adopted is the same as in chapters II and III; that is, (i) we express the jump Markov process in terms of the initial value, the jump times and the values taken after the jumps, instead of its instantaneous values, and then (ii) we apply Bayes' rule to obtain the a posteriori probability distribution of the jump process. The optimal estimate is given in terms of the a posteriori probability distribution of the jump process and the continuous-discrete-Kalman-filter estimates corresponding to the admissible values of the jump process. The resultant optimal estimator algorithm is, however, infinite dimensional, so that a feasible approximate estimator algorithm is presented for the practical implementation.

In section 4.2, the state estimation problem for a linear continuous-discrete system with a jump process is precisely formulated. The minimum-variance estimator algorithm is derived in section 4.3, and the approximate estimator algorithm is proposed in section 4.4 for the practical implementation. Simulation studies are also carried out, in section 4.5, to demonstrate the feasibility of the approximate estimator algorithm.

4.2 Statement of Problem

Consider the system represented by a stochastic differential equation:

$$dx(t) = F(t,\gamma(t))x(t)dt + G(t,\gamma(t))u(t)dt + D(t,\gamma(t))dw(t), \quad 0 \leq t, \tag{2.1}$$

and let the discrete observation be given by

$$y(t_k) = H(t_k,\gamma(t_k))x(t_k) + E(t_k,\gamma(t_k))v(t_k), \quad k = 1, 2, \ldots, \quad 0 \leq t_1 < t_2 < \cdots, \tag{2.2}$$

where

$x(t)$: an $n \times 1$ state vector at time t;

$y(t_k)$: a $p \times 1$ observation vector at discrete time t_k;

$u(t)$: a $q \times 1$ deterministic input vector;

$F(\cdot,\cdot)$: an $n \times n$ state transition vector;

$G(\cdot,\cdot)$: an $n \times q$ gain matrix;

$H(\cdot,\cdot)$: a $p \times n$ observation matrix;

$w(t)$: an $r \times 1$ standard Wiener process;

$v(t_k)$: a $p \times 1$ zero-mean independent Gaussian sequence with unit variance matrix;

$D(\cdot,\cdot)$: an $n \times r$ matrix;

$E(\cdot,\cdot)$: a $p \times p$ nonsingular matrix;

and

$\gamma(t)$: a right-continuous jump Markov process taking on values of $1, 2, \ldots, M$ with transition rates $q_{ij}(t)$ $(i,j=1,2,\ldots,M;\ i \neq j)$.

It is assumed that $x(0)$ is Gaussian with mean vector $\hat{x}(0)$ and covariance matrix $\hat{P}(0)$ and that stochastic processes $w(t)$, $v(t_k)$, $\gamma(t)$ and $x(0)$ are mutually independent.

The problem is to find the minimum-variance estimate $\hat{x}(t|t_k)$ of the state $x(t)$ based upon observations $Y^{t_k} \triangleq \{y(t_1), y(t_2), \ldots, y(t_k)\}$, where $t_k \leqq t < t_{k+1}$.

4.3 Optimal Estimator Algorithm

In this section, we shall derive the minimum-variance estimator algorithm for continuous-discrete system (2.1) and (2.2). The minimum-variance estimate $\hat{x}(t|t_k)$ of the state $x(t)$ given observations Y^{t_k} for $t_k \leq t < t_{k+1}$ is given by the conditional expectation:

$$\hat{x}(t|t_k) = E\{x(t)|Y^{t_k}\}. \tag{3.1}$$

By the smoothing property of conditional expectations (Doob, 1953), equation (3.1) is expressed as

$$\hat{x}(t|t_k) = E\{E\{x(t)|\gamma(s),0\leq s\leq t,Y^{t_k}\}|Y^{t_k}\}. \tag{3.2}$$

In (3.2), the inner conditional expectation $E\{x(t)|\gamma(s),0\leq s\leq t, Y^{t_k}\}$ can be obtained by the usual continuous-discrete-Kalman-filter algorithm (Jazwinski, 1970), because the values of the jump process $\{\gamma(s),0\leq s\leq t\}$ are specified as conditioning; whereas the outer conditional expectation must be performed over the random function $\{\gamma(s),0\leq s\leq t\}$. In order to overcome this difficulty, let us define:

τ_n = the random time that the nth jump of the process $\gamma(t)$ occurs; $(3.3)_1$

$$j_n = \gamma(\tau_n + 0), \tag{3.3$_2$}$$

as in chapters II and III. By the above definition, the random function $\{\gamma(s), 0 \leq s \leq t\}$ can be expressed in terms of the random sequence $\{\gamma(0), \tau_1, j_1, \tau_2, j_2, \dots\}$, which makes it possible for us to obtain the optimal estimate $\hat{x}(t|t_k)$ given by (3.2) in the explicit form as follows.

Theorem 4.1 (Optimal Estimator Algorithm for Continuous-Discrete Systems with Jump Parameter)

The minimum-variance estimate $\hat{x}(t|t_k)$ $(t_k \leq t < t_{k+1})$ for continuous-discrete system (2.1) and (2.2) is given by

$$\begin{aligned}\hat{x}(t|t_k) = & \sum_{i_0=1}^{M} \hat{x}(t|t_k, \{\gamma(0)=i_0, \tau_1>t\}) P(\gamma(0)=i_0, \tau_1>t | Y^{t_k}) \\ & + \sum_{n=1}^{\infty} \sum_{i_0=1}^{M} \sum_{\substack{i_1=1 \\ i_1 \neq i_0}}^{M} \cdots \sum_{\substack{i_n=1 \\ i_n \neq i_{n-1}}}^{M} \int_0^t \int_{s_1}^t \cdots \int_{s_{n-1}}^t \hat{x}(t|t_k, \{\gamma(0)=i_0, \tau^n=s^n, j^n=i^n, \tau_{n+1}>t\}) \\ & \times P(\gamma(0)=i_0, \tau^n \epsilon ds^n, j^n=i^n, \tau_{n+1}>t | Y^{t_k}),\end{aligned} \tag{3.4}$$

where $\tau^n = (\tau_1, \tau_2, \dots, \tau_n)$, $s^n = (s_1, s_2, \dots, s_n)$, $j^n = (j_1, j_2, \dots, j_n)$, $i^n = (i_1, i_2, \dots, i_n)$ and

$$\hat{x}(t|t_k, \{\gamma(s), 0 \leq s \leq t\}) \triangleq E\{x(t)|\gamma(s), 0 \leq s \leq t, Y^{t_k}\}. \tag{3.5}$$

Also, the a posteriori probabilities $P(\gamma(0)=i_0, \tau_1>t|Y^{t_k})$ and

$P(\gamma(0)=i_0,\tau^n \varepsilon ds^n, j^n=i^n, \tau_{n+1}>t|Y^{t_k})$ appearing in the right-hand side of (3.4) are, respectively, given by

$$P(\gamma(0)=i_0,\tau_1>t|Y^{t_k}) = \frac{p(y(t_k)|\gamma(0)=i_0,\tau_1>t,Y^{t_{k-1}})P(\gamma(0)=i_0,\tau_1>t|Y^{t_{k-1}})}{p(y(t_k)|Y^{t_{k-1}})} \tag{3.6}$$

and

$$P(\gamma(0)=i_0,\tau^n \varepsilon ds^n,j^n=i^n,\tau_{n+1}>t|Y^{t_k})$$

$$= \frac{p(y(t_k)|\gamma(0)=i_0,\tau^n=s^n,j^n=i^n,\tau_{n+1}>t,Y^{t_{k-1}})}{p(y(t_k)|Y^{t_{k-1}})} *$$

$$* \frac{P(\gamma(0)=i_0,\tau^n \varepsilon ds^n,j^n=i^n,\tau_{n+1}>t|Y^{t_{k-1}})}{} . \tag{3.7}$$

Here, the conditional probability density function $p(y(t_k)|\gamma(s), 0\leqq s\leqq t,Y^{t_{k-1}})$ is Gaussian with

$$\text{mean} = H(t_k,\gamma(t_k))\hat{x}(t_k|t_{k-1},\{\gamma(s),0\leqq s\leqq t_k\}) \tag{3.8$_1$}$$

$$\text{cov} = H(t_k,\gamma(t_k))\hat{P}(t_k|t_{k-1},\{\gamma(s),0\leqq s\leqq t_k\})H'(t_k,\gamma(t_k))$$
$$+ E(t_k,\gamma(t_k))E'(t_k,\gamma(t_k)), \tag{3.8}$$

where

$$\hat{P}(t_k|t_{k-1},\{\gamma(s),0\leqq s\leqq t_k\}) \triangleq \mathrm{Cov}\{x(t_k)|\gamma(s),0\leqq s\leqq t_k,Y^{t_{k-1}}\}. \tag{3.9}$$

Moreover, the conditional probability density function $p(y(t_k)|Y^{t_{k-1}})$ appearing in (3.6) and (3.7) is given by

$$p(y(t_k)|Y^{t_{k-1}})$$
$$= \sum_{i_0=1}^{M} p(y(t_k)|\gamma(0)=i_0,\tau_1>t_k,Y^{t_{k-1}})P(\gamma(0)=i_0,\tau_1>t_k|Y^{t_{k-1}})$$
$$+ \sum_{n=1}^{\infty} \sum_{i_0=1}^{M} \cdots \sum_{\substack{i_n=1 \\ i_n \neq i_{n-1}}}^{M} \int_0^{t_k}\int_{s_1}^{t_k}\cdots\int_{s_{n-1}}^{t_k} p(y(t_k)|\gamma(0)=i_0,\tau^n=s^n,j^n=i^n,\tau_{n+1}>t_k,Y^{t_{k-1}})$$
$$\times P(\gamma(0)=i_0,\tau^n \epsilon ds^n,j^n=i^n,\tau_{n+1}>t_k|Y^{t_{k-1}}). \qquad (3.10)$$

The conditional estimate $\hat{x}(t|t_k,\{\gamma(s),0\leqq s\leqq t\})$ defined by (3.5) and the associated covariance matrix $\hat{P}(t_k|t_{k-1};\{\gamma(s),0\leqq s\leqq t_k\})$ defined by (3.9) are obtained by the usual continuous-discrete-Kalman-filter algorithm, using the specified values of $\gamma(s)$. The initial conditions are given as follows:

$$E\{x(0)\} = \hat{x}(0), \quad Cov\{x(0)\} = \hat{P}(0),$$
$$P(\gamma(0)=i_0,\tau_1>t) = P(\gamma(0)=i_0)\exp\left\{-\int_0^t q_{i_0}(s)ds\right\}, \qquad (3.11)_1$$

and

$$P(\gamma(0)=i_0,\tau^n \epsilon ds^n,j^n=i^n,\tau_{n+1}>t)$$
$$= P(\gamma(0)=i_0)\exp\left\{\int_0^{s_1} q_{i_0}(s)ds - \cdots - \int_{s_n}^{t} q_{i_n}(s)ds\right\}$$
$$\times q_{i_0,i_1}(s_1)\cdots q_{i_{n-1},i_n}(s_n)ds_n\cdots ds_1, \qquad (3.11)_2$$

where

$$q_i(s) = \sum_{\substack{j=1 \\ j \neq i}}^{M} q_{i,j}(s).$$

Proof of Theorem 4.1

By definition (3.3), equation (3.2) can be expressed as

$$\hat{x}(t|t_k) = E\{E\{x(t)|\gamma(0),\tau_1,j_1,\tau_2,j_2,\ldots,Y^{t_k}\}|Y^{t_k}\}.$$

From this equation, we can easily show equation (3.4). Equations (3.6) and (3.7) are obtained by applying Bayes' rule (Ho and Lee, 1964). Also, by the similar procedure as in deriving (3.4), we get equation (3.10). Equation (3.11) is obtained from the property of the jump Markov process (Gikhman and Skorokhod, 1969).

Remark : The optimal estimator algorithm presented above is not generally feasible because the summation of the infinite number of terms must be carried out in (3.4) and (3.10). Thus the optimal estumator algorithm is not given in closed form, so that a feasible approximate estimator algorithm will be proposed in the following section for practical implementation.

4.4 Approximate Estimator Algorithm

In this section, we shall present a feasible approximate estimator algorithm by the similar approach as adopted in section 3.4.1, chapter III. We assume for simplicity that the jump process $\gamma(t)$ under consideration is stationary and that the sampling times for observations are equally spaced. We shall present an approximate estimator algorithm under the following assumption.

Assumption : The conditional probability density function $p(x(t_k)|Y^{t_k})$ of state $x(t_k)$ given observations Y^{t_k} is Gaussian with

$$\text{mean} = x^*(t_k|t_k) \tag{4.1$_1$}$$

$$\text{cov} = P^*(t_k|t_k) \tag{4.1$_2$}$$

for all $k > 0$.

For a sufficiently small positive number ε, choose a positive integer N_0 such that if $N \geqq N_0$ the jump process $\gamma(t)$ does not jump more than N times between the two successive sampling times with probability greater than $1 - \varepsilon$. Then, regarding t_k as the initial time and assuming that the jump process $\gamma(t)$ jumps at most N_0 times for $t_k \leq t \leq t_{k+1}$, we obtain from Assumption (4.1) an approximate estimate $x^*(t|t_k)$ $(t_k<t<t_{k+1})$ as follows.

Approximate Estimator Algorithm

The approximate estimate $x^*(t|t_k)$ $(t_k<t<t_{k+1})$ of state $x(t)$ given observations Y^{t_k} for system (2.1) and (2.2) is given by

$$x^*(t|t_k) = \sum_{i_0=1}^{M} x^*(t|t_k,\{\gamma(t_k)=i_0,\overline{\tau}_1>t\})P(\overline{\tau}_1>t|\gamma(t_k)=i_0) \times P(\gamma(t_k)=i_0|I^*(t_k))$$

$$+ \sum_{n=1}^{N_0} \sum_{i_0=1}^{M} \cdots \sum_{\substack{i_n=1 \\ i_n \neq i_{n-1}}}^{M} \int_{t_k}^{t} \int_{s_1}^{t} \cdots \int_{s_{n-1}}^{t} x^*(t|t_k,\{\gamma(t_k)=i_0,\overline{\tau}^n=s^n,\overline{j}^n=i^n,\overline{\tau}_{n+1}>t\})$$

$$\times P(\overline{\tau}^n \epsilon ds^n,\overline{j}^n=i^n,\overline{\tau}_{n+1}>t|\gamma(t_k)=i_0)$$

$$\times P(\gamma(t_k)=i_0|I^*(t_k)) , \qquad (4.2)$$

where

$$I^*(t_k) \triangleq \{x^*(t_k|t_k),P^*(t_k|t_k)\} \qquad (4.3)$$

and

$$x^*(t|t_k,\{\gamma(s),t_k\leq s\leq t\}) \triangleq E\{x(t)|\gamma(s),t_k\leq s\leq t,I^*(t_k)\}. \qquad (4.4)$$

Here, $\overline{\tau}_n$ and $\overline{j}_n$ appearing in (4.2) are defined as follows:

$\bar{\tau}_n$ = the random time that the nth jump of $\gamma(t)$ occurs after time t_k;

$\bar{j}_n = \gamma(\bar{\tau}_n+0)$.

Also, the approximate estimate $x^*(t_{k+1}|t_{k+1})$ is given by

$$
\begin{aligned}
&x^*(t_{k+1}|t_{k+1}) \\
&= \sum_{i_0=1}^{M} x^*(t_{k+1}|t_{k+1},\{\gamma(t_k)=i_0,\bar{\tau}_1>t_{k+1}\}) \\
&\qquad \times P(\gamma(t_k)=i_0,\bar{\tau}_1>t_{k+1}|I^*(t_k),y(t_{k+1})) \\
&+ \sum_{n=1}^{N_0} \sum_{i_0=1}^{M} \cdots \sum_{\substack{i_n=1 \\ i_n\neq i_{n-1}}}^{M} \int_{t_k}^{t_{k+1}} \int_{s_1}^{t_{k+1}} \cdots \int_{s_{n-1}}^{t_{k+1}} \\
&\qquad x^*(t_{k+1}|t_{k+1},\{\gamma(t_k)=i_0,\bar{\tau}^n=s^n,\bar{j}^n=i^n,\bar{\tau}_{n+1}>t_{k+1}\}) \\
&\qquad \times P(\gamma(t_k)=i_0,\bar{\tau}^n \varepsilon ds^n,\bar{j}^n=i^n,\bar{\tau}_{n+1}>t_{k+1}|I^*(t_k),y(t_{k+1})),
\end{aligned} \tag{4.5}
$$

where

$$
\begin{aligned}
&x^*(t_{k+1}|t_{k+1},\{\gamma(s),t_k\leq s\leq t_{k+1}\}) \\
&= E\{x(t_{k+1})|\gamma(s),t_k\leq s\leq t_{k+1},I^*(t_k),y(t_{k+1})\}.
\end{aligned} \tag{4.6}
$$

The associated approximate error covariance matrix $P^*(t_{k+1}|t_{k+1})$ is given by

$$P^*(t_{k+1}|t_{k+1})$$
$$= \sum_{i_0=1}^{M} \overline{P}(t_{k+1},\{\gamma(t_k)=i_0,\overline{\tau}_1>t_{k+1}\})P(\gamma(t_k)=i_0,\overline{\tau}_1>t_{k+1}|I^*(t_k),y(t_{k+1}))$$
$$+ \sum_{n=1}^{N_0} \sum_{i_0=1}^{M} \cdots \sum_{\substack{i_n=1 \\ i_n\neq i_{n-1}}}^{M} \int_{t_k}^{t_{k+1}}\int_{s_1}^{t_{k+1}}\cdots\int_{s_{n-1}}^{t_{k+1}}$$
$$\overline{P}(t_{k+1},\{\gamma(t_k)=i_0,\overline{\tau}^n=s^n,\overline{j}^n=i^n,\overline{\tau}_{n+1}>t_{k+1}\})$$
$$\times P(\gamma(t_k)=i_0,\overline{\tau}^n\epsilon ds^n,\overline{j}^n=i^n,\overline{\tau}_{n+1}>t_{k+1}|I^*(t_k),y(t_{k+1})), \tag{4.7}$$

where

$$\overline{P}(t_{k+1},\{\gamma(s),t_k\leq s\leq t_{k+1}\})$$
$$\triangleq P^*(t_{k+1}|t_{k+1},\{\gamma(s),t_k\leq s\leq t_{k+1}\})$$
$$+ [x^*(t_{k+1}|t_{k+1}) - x^*(t_{k+1}|t_{k+1},\{\gamma(s),t_k\leq s\leq t_{k+1}\})]$$
$$\times [x^*(t_{k+1}|t_{k+1}) - x^*(t_{k+1}|t_{k+1},\{\gamma(s),t_k\leq s\leq t_{k+1}\})]' \tag{4.8}$$

and

$$P^*(t_{k+1}|t_{k+1},\{\gamma(s),t_k\leq s\leq t_{k+1}\})$$
$$\triangleq \mathrm{Cov}\{x(t_{k+1})|\gamma(s),t_k\leq s\leq t_{k+1},I^*(t_k),y(t_{k+1})\}. \tag{4.9}$$

The a posteriori probabilities $P(\gamma(t_k)=i_0,\overline{\tau}_1>t_{k+1}|I^*(t_k),y(t_{k+1}))$ and $P(\gamma(t_k)=i_0,\overline{\tau}^n\epsilon ds^n,\overline{j}^n=i^n,\overline{\tau}_{n+1}>t_{k+1}|I^*(t_k),y(t_{k+1}))$ appearing in (4.5) and (4.7) are given as follows.

$$P(\gamma(t_k)=i_0,\bar{\tau}_1>t_{k+1}|I^*(t_k),y(t_{k+1}))$$

$$=\frac{p(y(t_{k+1})|\gamma(t_k)=i_0,\bar{\tau}_1>t_{k+1},I^*(t_k))P(\bar{\tau}_1>t_{k+1}|\gamma(t_k)=i_0)}{p(y(t_{k+1})|I^*(t_k))} *$$

$$* \frac{P(\gamma(t_k)=i_0|I^*(t_k))}{} \qquad (4.10)$$

and

$$P(\gamma(t_k)=i_0,\bar{\tau}^n \varepsilon ds^n,\bar{j}^n=i^n,\bar{\tau}_{n+1}>t_{k+1}|I^*(t_k),y(t_{k+1}))$$

$$=\frac{p(y(t_{k+1})|\gamma(t_k)=i_0,\bar{\tau}^n=s^n,\bar{j}^n=i^n,\bar{\tau}_{n+1}>t_{k+1},I^*(t_k))}{p(y(t_{k+1})|I^*(t_k))} *$$

$$* \frac{P(\bar{\tau}^n \varepsilon ds^n,\bar{j}^n=i^n,\bar{\tau}_{n+1}>t_{k+1}|\gamma(t_k)=i_0)P(\gamma(t_k)=i_0|I^*(t_k))}{}, \qquad (4.11)$$

where from Assumption (4.1) the probability density fuction $p(y(t_{k+1})|\gamma(s),t_k\leqq s\leqq t_{k+1},I^*(t_k))$ is Gaussian with

$$\text{mean} = H(t_{k+1},\gamma(t_{k+1}))x^*(t_{k+1}|t_k,\{\gamma(s),t_k\leqq s\leqq t_{k+1}\}) \qquad (4.12)_1$$

$$\begin{aligned}\text{cov} = {} & H(t_{k+1},\gamma(t_{k+1}))P^*(t_{k+1}|t_k,\{\gamma(s),t_k\leqq s\leqq t_{k+1}\}) \\ & \times H'(t_{k+1}, (t_{k+1})) \\ & + E(t_{k+1},\gamma(t_{k+1}))E'(t_{k+1},\gamma(t_{k+1})).\end{aligned} \qquad (4.12)_2$$

The probability density function $p(y(t_{k+1})|I^*(t_k))$ appearing in (4.10) and (4.11) is given by equation (4.7) which results from replacing $\overline{P}(t_{k+1},\{\gamma(s),t_k\leq s\leq t_{k+1}\})$ by $p(y(t_{k+1})|\gamma(s), t_k\leq s\leq t_{k+1},I^*(t_k))$. Moreover, the conditional probabilities $P(\overline{\tau}_1>t|\gamma(t_k)=i_0)$ and $P(\overline{\tau}^n\varepsilon ds^n,\overline{j}^n=i^n,\overline{\tau}_{n+1}>t|\gamma(t_k)=i_0)$ are expressed in terms of transition rates q_{ij} as

$$P(\overline{\tau}_1>t|\gamma(t_k)=i_0) = \exp\{-q_{i_0}(t-t_k)\} \tag{4.13}$$

and

$$\begin{aligned} &P(\overline{\tau}^n\varepsilon ds^n,\overline{j}^n=i^n,\overline{\tau}_{n+1}>t|\gamma(t_k)=i_0) \\ &= \exp\{-q_{i_0}(s_1-t_k) - q_{i_1}(s_2-s_1) - \cdots - q_{i_n}(t-s_n)\} \\ &\quad\times q_{i_0,i_1}q_{i_1,i_2}\cdots q_{i_{n-1},i_n}ds_n\cdots ds_2ds_1 \, . \end{aligned} \tag{4.14}$$

Furthermore, the a posteriori probabilities $P(\gamma(t_k)=r|I^*(t_k))$ $(r=1,2,\ldots,M)$ appearing in (4.2) are given as follows.

$$\begin{aligned} &P(\gamma(t_k)=r|I^*(t_k)) \\ &= P(\gamma(t_{k-1})=r,\overline{\tau}_1>t_k|I^*(t_{k-1}),y(t_k)) \\ &+ \sum_{\substack{i_0=1\\ i_0\neq r}}^{M}\int_{t_{k-1}}^{t_k} P(\gamma(t_{k-1})=i_0,\overline{\tau}_1\varepsilon ds_1,\overline{j}_1=r,\overline{\tau}_2>t_k|I^*(t_{k-1}),y(t_k)) \\ &+ \sum_{n=2}^{N_0}\sum_{i_0=1}^{M}\sum_{\substack{i_1=1\\ i_1\neq i_0}}^{M}\cdots\sum_{\substack{i_{n-1}=1\\ i_{n-1}\neq\{i_{n-2},r\}}}^{M}\int_{t_{k-1}}^{t_k}\int_{s_1}^{t_k}\cdots\int_{s_{n-1}}^{t_k} \\ &\quad\times P(\gamma(t_{k-1})=i_0,\overline{\tau}^n\varepsilon ds^n,\overline{j}^{n-1}=i^{n-1},\overline{j}_n=r,\overline{\tau}_{n+1}>t_k|I^*(t_{k-1}),y(t_k)). \end{aligned} \tag{4.15}$$

Equations (4.2)-(4.15) completely specify the approximate estimator algorithm

Derivation of Approximate Estimator Algorithm

By the smoothing property of conditional expectations, the approximate estimate $x^*(t|t_k)$ $(t_k<t<t_{k+1})$ of state $x(t)$ given Y^{t_k} is expressed as

$$\begin{aligned} x^*(t|t_k) &= E\{E\{x(t)|\gamma(s),t_k\leq s\leq t,I^*(t_k)\}|I^*(t_k)\} \\ &= E\{E\{x(t)|\gamma(t_k),\overline{\tau}^{N_0},\overline{j}^{N_0},I^*(t_k)\}|I^*(t_k)\}, \quad (4.16) \end{aligned}$$

where we assume that the process $\gamma(s)$ $(t_k\leq s\leq t_{k+1})$ jumps at most N_0 times. From (4.16), we can easily obtain (4.2). Similarly, the approximate estimate $x^*(t_{k+1}|t_{k+1})$ is expressed as

$$\begin{aligned} & x^*(t_{k+1}|t_{k+1}) \\ &= E\{E\{x(t_{k+1})|\gamma(t_k),\overline{\tau}^{N_0},\overline{j}^{N_0},I^*(t_k),y(t_{k+1})\}|I^*(t_k),y(t_{k+1})\} \end{aligned}$$

which results in (4.5).

Furthermore, the associated approximate error covariance matrix $P^*(t_{k+1}|t_{k+1})$ is given by

$$\begin{aligned} P^*(t_{k+1}|t_{k+1}) = E\{&[x(t_{k+1}) - x^*(t_{k+1}|t_{k+1})] \\ &\times [x(t_{k+1}) - x^*(t_{k+1}|t_{k+1})]'|I^*(t_k),y(t_{k+1})\} \end{aligned}$$

$$= E\Big\{P^*(t_{k+1}|t_{k+1},\{\gamma(t_k),\bar{\tau}^{N_0},\bar{j}^{N_0}\})$$
$$+ [x^*(t_{k+1}|t_{k+1}) - x^*(t_{k+1}|t_{k+1},\{\gamma(t_k),\bar{\tau}^{N_0},\bar{j}^{N_0}\})]$$
$$\times [x^*(t_{k+1}|t_{k+1})$$
$$- x^*(t_{k+1}|t_{k+1},\{\gamma(t_k),\bar{\tau}^{N_0},\bar{j}^{N_0}\})]'|I^*(t_k),y(t_{k+1})\Big\}$$

which leads to equations (4.7)-(4.9).

By applying Bayes' rule (Ho and Lee, 1964), we can easily get the a posteriori probabilities $P(\gamma(t_k)=i_0,\bar{\tau}_1>t_{k+1}|I^*(t_k),y(t_{k+1}))$ and $P(\gamma(t_k)=i_0,\bar{\tau}^n\varepsilon ds^n,\bar{j}^n=i^n,\bar{\tau}_{n+1}>t_{k+1}|I^*(t_k),y(t_{k+1}))$, which are given by (4.10) and (4.11), respectively. Equations (4.13) and (4.14) are obtained from the properties of the jump Markov process (Gikhman and Skorokhod, 1969) and the assumption of the stationarity of the jump process. Equation (4.15) follows from

$$P(\gamma(t_k)=r|I^*(t_k))$$
$$= E\{P(\gamma(t_k)=r|\gamma(s),t_{k-1}\leq s<t_k,I^*(t_k))|I^*(t_k)\}.$$

This completes the derivation of the approximate estimator algorithm.

Remark 1 : It is to be noted that the positive number ε, which appears in the definition of integer N_0, should be determined according to the capacity of the computer memory and the desired accuracy of the approximate estimate.

Remark 2 : The approximate estimator algorithm proposed here can also be employed for the case: when the optimal estimator algorithm is not feasible due to the insufficiency of the capacity of the computer memory, though the number of the jumps of the process $\gamma(t)$ is bounded with probability 1 so that the summation of a finite number of terms has only to be carried out in the optimal algorithm.

4.5 Numerical Example

We shall show an illustrative example in order to demonstrate the feasibility of the proposed approximate estimator algorithm. Let us consider the following scalar system:

$$dx(t) = f(\gamma(t))x(t)dt + g(\gamma(t))u(t)dt + \delta dw(t) \tag{5.1}$$

$$y(t_k) = x(t_k) + ev(t_k), \tag{5.2}$$

where $\gamma(t)$ is a stationary jump Markov process taking on values of 1, 2, 3. We assume that only the transitions from 1 to 2 and from 2 to 3 are possible for the jump process $\gamma(t)$.

This means that $q_{1,3} = q_{2,1} = q_{3,2} = q_{3,1} = 0$. It is also assumed that $\gamma(0) = 1$ with probability 1.

Equations (4.2) and (4.5) for system (5.1) and (5.2) become

$$\begin{aligned} x^*(t|t_k) = & \sum_{i_0=1}^{3} x^*(t|t_k,\{\gamma(t_k)=i_0,\overline{\tau}_1>t\})P(\overline{\tau}_1>t|\gamma(t_k)=i_0) \\ & \qquad \times P(\gamma(t_k)=i_0|I^*(t_k)) \\ & + \sum_{i_0=1}^{2}\int_{t_k}^{t} x^*(t|t_k,\{\gamma(t_k)=i_0,\overline{\tau}_1=s_1,\overline{j}_1=i_0+1,\overline{\tau}_2>t\}) \\ & \qquad \times P(\overline{\tau}_1\varepsilon ds_1,\overline{j}_1=i_0+1,\overline{\tau}_2>t|\gamma(t_k)=i_0) \\ & \qquad \times P(\gamma(t_k)=i_0|I^*(t_k)) \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} x^*(t_{k+1}|t_{k+1}) = & \sum_{i_0=1}^{3} x^*(t_{k+1}|t_{k+1},\{\gamma(t_k)=i_0,\overline{\tau}_1>t_{k+1}\}) \\ & \qquad \times P(\gamma(t_k)=i_0,\overline{\tau}_1>t_{k+1}|I^*(t_k),y(t_{k+1})) \\ & + \sum_{i_0=1}^{2}\int_{t_k}^{t_{k+1}} x^*(t_{k+1}|t_{k+1},\{\gamma(t_k)=i_0,\overline{\tau}_1=s_1,\overline{j}_1=i_0+1,\overline{\tau}_2>t_{k+1}\}) \\ & \qquad \times P(\gamma(t_k)=i_0,\overline{\tau}_1\varepsilon ds_1,\overline{j}_1=i_0+1,\overline{\tau}_2>t_{k+1}|I^*(t_k),y(t_{k+1})) \end{aligned} \tag{5.4}$$

where ε and N_0 defined in the previous section are chosen to be 0.018 and 1, respectively.

Also, the a posteriori probabilities $P(\gamma(t_k)=i_0|I^*(t_k))$ $(i_0=1,2,3)$ are given by

$$P(\gamma(t_k)=1|I^*(t_k)) = P(\gamma(t_{k-1})=1,\overline{\tau}_1>t_k|I^*(t_{k-1}),y(t_k)) \tag{5.5}$$

$$P(\gamma(t_k)=i \mid I^*(t_k))$$
$$= P(\gamma(t_{k-1})=i, \bar{\tau}_1 > t_k \mid I^*(t_{k-1}), y(t_k))$$
$$+ \int_{t_{k-1}}^{t_k} P(\gamma(t_{k-1})=i-1, \bar{\tau}_1 \varepsilon ds_1, \bar{j}_1 = i, \bar{\tau}_2 > t_k \mid I^*(t_{k-1}), y(t_k))$$
$$\text{for} \quad i = 2, 3. \tag{5.6}$$

Simulation studies are carried out over the time interval [0,1] using the following set of numerical values:

$f(1) = -4.5$, $f(2) = 0$, $f(3) = 5.0$, $g(1) = 1.0$,

$\delta = 1.414$, $e = 1.0$, $q_{1,2} = 2.0$, $q_{2,3} = 2.0$,

$\hat{x}(0) = 5$, $\hat{P}(0) = 10$, $u(t) = 1.0$, $x(0) = 10$,

$\gamma(0) = 1$, $t_k = 0.1 \times k$,

and we set the time increment $\Delta t = 0.01$.

Fig. 4.1 shows the sample behavior of the state $x(t)$ and the approximate estimate $x^*(t|t_k)$ together with the estimate $\bar{x}(t|t_k)$ by the continuous-discrete Kalman filter given the true realization of the jump process $\gamma(t)$. The corresponding behavior of the jump process $\gamma(t)$ and the approximate a posteriori probabilities $P(\gamma(t)=i \mid I^*(t_k))$ $(i=1,2,3;\ t_k \leq t < t_{k+1},\ k=1,2,\ldots,9)$ is displayed in Fig. 4.2. We may observe from Fig. 4.1 that the proposed approximate estimate is nearly optimal. It should be noted that, using the approximate a posteriori probabilities $P(\gamma(t)=i|I^*(t_k))$ $(i=1,2,3)$, we can perform detection and estimation of the jump process $\gamma(t)$.

4.6 Concluding Remarks

The minimum-variance estimator algorithm is derived for a class of continuous-discrete systems with a jump process. The optimal estimator algorithm is, however, infeasible, so that an approximate estimator algorithm is proposed for practical implementation. Simulation studies are also carried out to demonstrate the feasibility of the proposed approximate estimator algorithm. It should be noted that by the similar procedure as taken in this chapter we can also obtain the optimal and the approximate estimator algorithms for the following continuous-discrete system :

$$dx(t) = F(t)x(t)dt + G(t)u(t)dt + Q(t)dw(t)$$

and

$$y(t_k) = \int_{t_{k-1}}^{t_k} H(s)x(s)ds + R(t_k)v(t_k),$$

where matrices $F(t)$, $G(t)$, $Q(t)$, $H(s)$ and $R(t)$ may be modulated by a jump process. The estimation problem for the above continuous-discrete system without jump parameter is treated in (Fujishige, 1975). The details are omitted here.

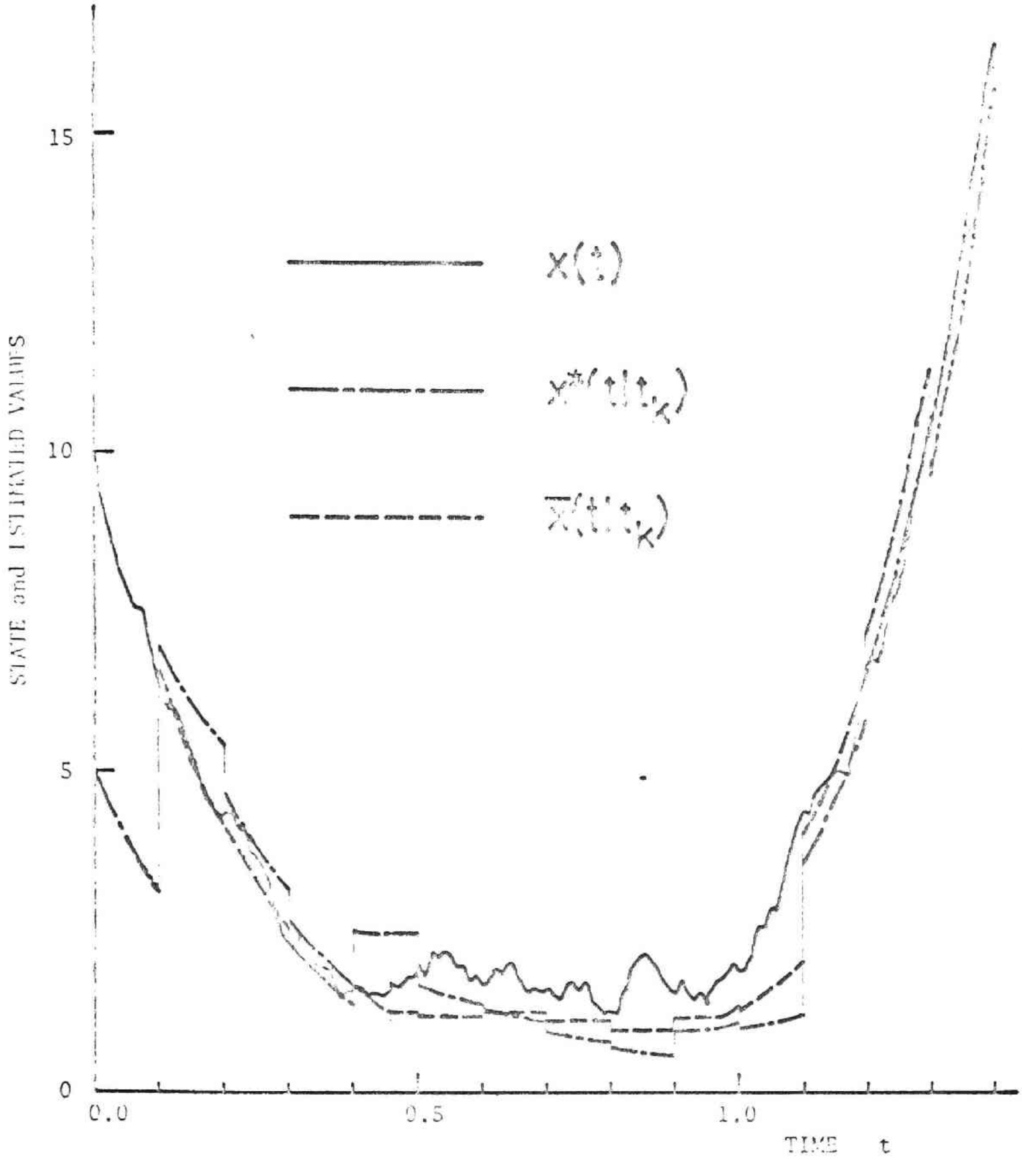

Fig. 4.1. Sample paths of the state $x(t)$, the approximate estimate $x^*(t|t_k)$ and the optimal estimate $\overline{x}(t|t_k)$ given the realization of $\gamma(t)$.

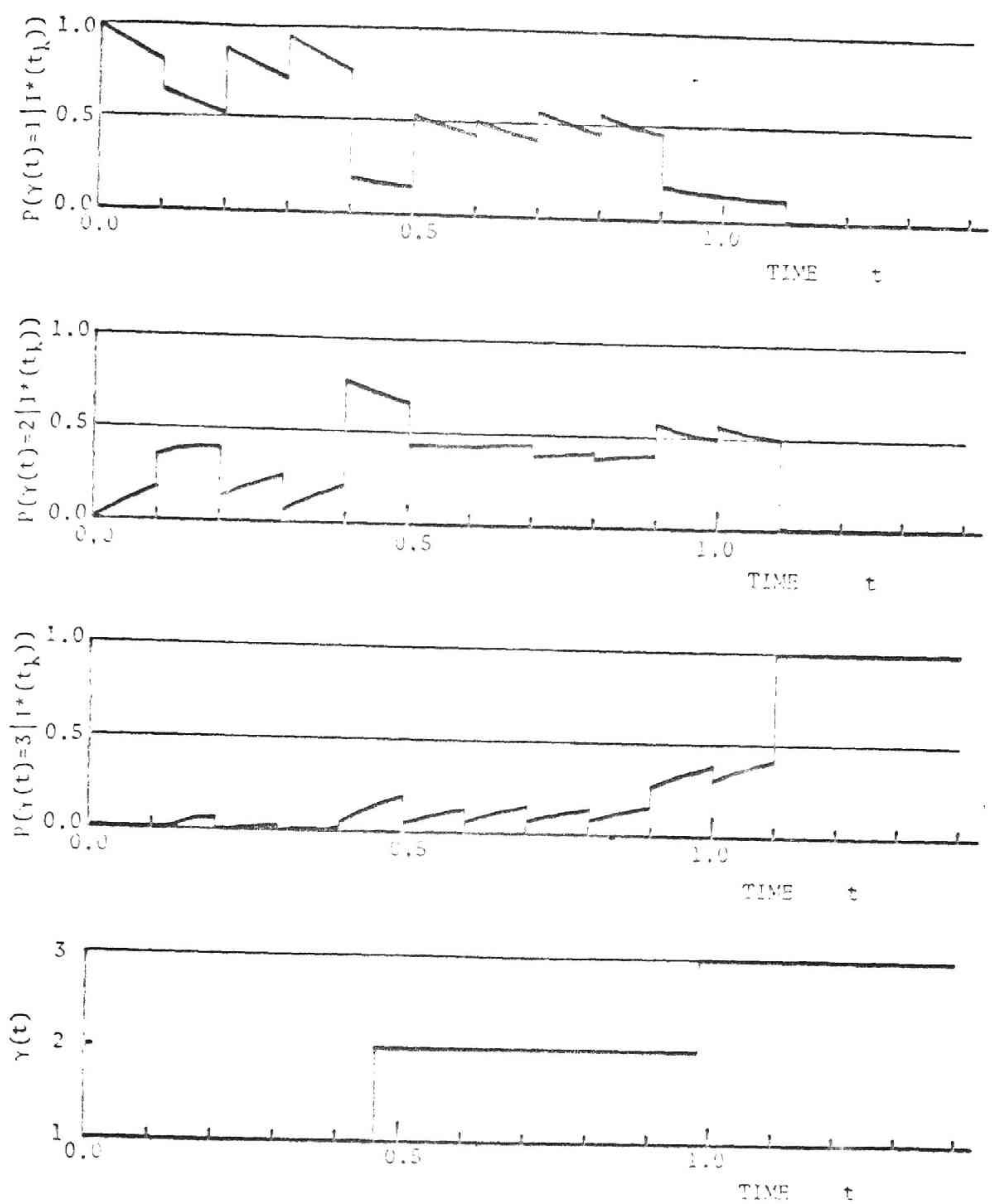

Fig. 4.2. A sample path of $\gamma(t)$, and the corresponding behavior of the approximate a posteriori probabilities $P(\gamma(t)=i|I^*(t_k))$ $(i=1,2,3)$.

CHAPTER V

STOCHASTIC CONTROL FOR LINEAR STOCHASTIC SYSTEM WITH JUMP PARAMETER

5.1 Introduction

In the previous chapters, we have considered estimation problems for linear discrete, continuous, and continuous-discrete systems with jump parameters. Usually, problems are not only to find out the optimal estimate of the state but also to determine the optimal control policy by employing the estimated value of the state. Thus, in this chapter, we consider the combined optimal estimation and control problems for linear stochastic systems with jump parameters.

In section 5.2, considered is the optimal control problem for linear discrete systems with switching environments, where characteristics of system- and measurement-noise processes change according to a Markov chain. The continuous-discrete counterpart is treated in section 5.3. A suboptimal control algorithm for discrete systems with a Markov chain is presented in section 5.4, where the system components are submitted to random failure. Here, an expected quadratic cost is taken as a performance criterion throughout this chapter. In sections 5.2 and 5.3, shown is a separation theorem that the optimal control input consists of two parts: (i) the optimal control input for linear-quadratic-Gaussian systems and (ii) its correction input due to the fact that the noise processes are non-white and have non-zero means.

5.2 Optimal Control for Discrete System with Switching Environment

We shall consider the optimal control problem for linear discrete systems with switching environments, where the characteristics of the system- and measurement-noise processes depend upon a Markov chain.

Consider the linear discrete system represented by

$$x(k+1) = F(k)x(k) + G(k)u(k) + a(k,\gamma(k+1)) + Q(k,\gamma(k+1))w(k), \tag{2.1}$$

and let the observation be given by

$$y(k) = H(k)x(k) + b(k,\gamma(k)) + R(k,\gamma(k))v(k), \tag{2.2}$$

where

$x(k)$: an $n \times 1$ state vector;

$y(k)$: a $p \times 1$ observation vector;

$u(k)$: an $r \times 1$ control input vector;

$F(k)$: an $n \times n$ state transition matrix;

$G(k)$: an $n \times r$ control gain matrix;

$H(k)$: a $p \times n$ observation matrix;

$a(k,\cdot)$: an $n \times 1$ vector;

$b(k,\cdot)$: a $p \times 1$ vector;

$Q(k,\cdot)$: an $n \times m$ matrix;

$R(k,\cdot)$: a $p \times p$ nonsingular matrix;

$w(k)$: an $m \times 1$ zero-mean Gaussian white noise sequence with unit variance Matrix;

$v(k)$: a $p \times 1$ zero-mean Gaussian white noise sequence with unit variance matrix;

$\gamma(k)$: a Markov chain taking on values of $1, 2, \ldots, M$ with transition probability matrix:

$$P = \left[p_{ij}(k) \right], \quad p_{ij}(k) \triangleq P(\gamma(k)=j \,|\, \gamma(k-1)=i),$$
$$i, j = 1, 2, \ldots, M,$$

and it is assumed that stochastic processes $w(k)$, $v(k)$, $\gamma(k)$ and $x(0)$ are mutually independent and that the initial state $x(0)$ is Gaussian.

The problem is to find out the optimal control policy which minimizes the expected quadratic cost:

$$J = E\left\{ \sum_{k=0}^{N} [x'(k+1)P(k+1)x(k+1) + u'(k)Q_1(k)u(k)] \right\}, \quad (2.3)$$

where $P(k)$ $(k=1,2,\ldots,N+1)$ are $n \times n$ symmetric nonnegative definite matrices and $Q_1(k)$ $(k=1,2,\ldots,N)$ are $r \times r$ symmetric positive definite matrices. Here, we assume that the admissible control $u(k)$ at time k is Y^k-measurable, where Y^k is a minimum σ-field generated by observations $\{y(0),y(1),\ldots,y(k)\}$.

The optimal control algorithm for system (2.1) and (2.2) with quadratic cost (2.3) is given by the following theorem.

Theorem 5.1 (Optimal Control Algorithm)

The optimal control input $u^*(k)$ for system (2.1) and (2.2) with criterion (2.3) is given by

$$u^*(k) = - K(k)\hat{x}(k|k) - L(k)\hat{w}^*(k|k), \tag{2.4}$$

where

$$\hat{x}(k|k) \triangleq E\{x(k)|Y^k\},$$

and

$$K(k) = [G'(k)P^*(k+1)G(k) + Q_1(k)]^{-1}G'(k)P^*(k+1)F(k) \tag{2.5}$$

$$L(k) = [G'(k)P^*(k+1)G(k) + Q_1(k)]^{-1}G'(k). \tag{2.6}$$

Here, $P^*(k+1)$ appearing in (2.5) and (2.6) is given by the backward equation:

$$P^*(k) = F'(k)P^*(k+1)F(k) + P(k) - P_0(k), \tag{2.7}$$

$$P_0(k) = F'(k)P^*(k+1)G(k)K(k) \tag{2.8}$$

and

$$P^*(N+1) = P(N+1), \quad k = N, N-1, \ldots, 0.$$

Also, $\hat{w}^*(k|k)$ is defined by

$$\hat{w}^*(k|k) \triangleq E\{w^*(k)|Y^k\} , \tag{2.9}$$

where

$$w^*(k) = P^*(k+1)\xi(k) + T(k+1)w^*(k+1) \qquad (2.10)_1$$

$$\xi(k) = a(k,\gamma(k+1)) + Q(k,\gamma(k+1))w(k) \qquad (2.10)_2$$

$$T(k) = F'(k) - F'(k)P^*(k+1)G(k)L(k) \tag{2.11}$$

and

$$w^*(N) = P(N+1)\xi(N). \tag{2.12}$$

Remark : Theorem 5.1 shows that a certainty equivalence (Patchel and Jacobs, 1971) holds for system (2.1) and (2.2) with criterion (2.3).

Proof of Theorem 5.1

We devide the observation process $y(k)$ into the component $y_1(k)$ functionally dependent upon the past control inputs $U^{k-1} \triangleq \{u(k-1), u(k-2), \ldots, u(0)\}$ and the component $y_2(k)$ functionally independent of the past control inputs U^{k-1}. That is, we consider processes $y_1(k)$ and $y_2(k)$ generated by

$$x_1(k+1) = F(k)x_1(k) + G(k)u(k), \quad x_1(0) = 0, \tag{2.13}$$

$$y_1(k) = H(k)x_1(k) \tag{2.14}$$

and

$$x_2(k+1) = F(k)x_2(k) + a(k,\gamma(k+1)) + Q(k,\gamma(k+1))w(k),$$

$$x_2(0) = x(0), \tag{2.15}$$

$$y_2(k) = H(k)x_2(k) + b(k,\gamma(k)) + R(k,\gamma(k))v(k), \tag{2.16}$$

respectively. From (2.13)-(2.16), we have

$$x(k) = x_1(k) + x_2(k) \tag{2.17}$$

and

$$y(k) = y_1(k) + y_2(k). \tag{2.18}$$

Here, $y_1(k)$ is completely determined by U^{k-1}, and $y_2(k)$ is functionally independent of U^{k-1} and is given by (2.18) in terms of $y(k)$ and $y_1(k)$. This means that

"the amount of information contained in $\{Y^k, U^{k-1}\}$ is equivalent to the one contained in $\{Y_2{}^k, U^{k-1}\}$", (2.19)

where $Y_2{}^k$ is a σ-field generated by $\{y_2(0), y_2(1), \ldots, y_2(k)\}$.

Now, we shall show the following lemma.

lemma : Estimation errors:

$$\tilde{x}(k|k) \triangleq x(k) - \hat{x}(k|k) \tag{2.20}$$

and

$$\tilde{\xi}(k|k) \triangleq \xi(k) - \hat{\xi}(k|k) \tag{2.21}$$

are functionally independent of the past control inputs U^{k-1}, where

$$\hat{x}(k|k) = E\{x(k)|U^{k-1},Y^k\} \tag{2.22}$$

$$\hat{\xi}(k|k) = E\{\xi(k)|U^{k-1},Y^k\} \tag{2.23}$$

and

$$\xi(k) = a(k,\gamma(k+1)) + Q(k,\gamma(k+1))w(k). \tag{2.24}$$

Proof of lemma : From proposition (2.19) and equations (2.13)-(2.18), we have

$$\tilde{x}(k|k) = x_2(k) - E\{x_2(k)|Y_2{}^k\} \tag{2.25}$$

and

$$\tilde{\xi}(k|k) = \xi(k) - E\{\xi(k)|Y_2{}^k\}, \tag{2.26}$$

because process $x_2(k)$ defined by (2.15) and process $\xi(k)$ defined by (2.24) are independent of the past control inputs U^{k-1}. The above lemma follows from (2.25) and (2.26).

The fact that the estimation errors $\tilde{x}(k|k)$ and $\tilde{\xi}(k|k)$ are functionally independent of the past control inputs U^{k-1} is a crucial point for Theorem 5.1 to hold.

By Bellman's principle of optimality (Bellman, 1957), the optimal control input $u^*(k)$ is given by $u(k)$ which minimizes the right-hand side of the following functional equation:

$$J_k^* = \min_{u(k)} E\Big\{x'(k+1)P(k+1)x(k+1) + u'(k)Q_1(k)u(k) + J_{k+1}^* \mid U^{k-1}, Y^k\Big\} \qquad (2.27)$$

and

$$J_{N+1}^* \equiv 0,$$

where $k = N, N-1, \ldots, 0$. The optimal control input $u^*(N)$ which minimizes (2.27) for $k = N$ is given by

$$u^*(N) = - K(N)\hat{x}(N|N) - L(N)\hat{w}^*(N|N), \qquad (2.28)$$

where $K(N)$, $L(N)$ and $\hat{w}^*(N|N)$ are defined by (2.5)-(2.12), and then J_N^* becomes

$$J_N^* = E\Big\{x'(N)[F'(N)P(N+1)F(N) - P_0(N)]x(N) + 2x'(N)T(N)w^*(N) + I(N) \mid U^{N-1}, Y^N\Big\}, \qquad (2.29)$$

where

$$I(N) = \xi'(N)P(N+1)\xi(N) - w^{*\prime}(N)G(N)L(N)w^*(N) + [P(N+1)F(N)\tilde{x}(N|N) + \tilde{w}^*(N|N)]'G(N)L(N) \times [P(N+1)F(N)\tilde{x}(N|N) + \tilde{w}^*(N|N)]. \qquad (2.30)$$

Here, $I(N)$ defined by (2.30) is functionally independent of U^{N-1} by the above lemma, because $\tilde{w}^*(N|N)$ is obtained by a linear transformation of $\tilde{\xi}(N|N)$ by definition (2.12), (2.21) and (2.24).

Suppose that for $k = n$ we have

$$J_n^* = E\Big\{x'(n)[F'(n)P^*(n+1)F(n) - P_0(n)]x(n) + 2x'(n)T(n)w^*(n) + I(n)\,|\,U^{n-1},Y^n\Big\}, \tag{2.31}$$

where $I(n)$ is functionally independent of U^{n-1}. Substituting (2.31) into (2.27) for $k = n-1$ and performing the minimization with respect to $u(n-1)$ yields the optimal control input $u^*(n-1)$ given by (2.4)-(2.12) for $k = n-1$; and as a result, J_{n-1}^* becomes

$$J_{n-1}^* = E\Big\{x'(n-1)[F'(n-1)P^*(n)F(n-1) - P_0(n-1)]x(n-1) + 2x'(n-1)T(n-1)w^*(n-1) + I(n-1)\,|\,U^{n-2},Y^{n-1}\Big\}, \tag{2.32}$$

where

$$\begin{aligned} I(n-1) = I(n) &+ \xi'(n-1)P^*(n)\xi(n-1) \\ &- w^{*\prime}(n-1)G(n-1)L(n-1)w^*(n-1) \\ &+ [P^*(n)F(n-1)\tilde{x}(n-1|n-1) + \tilde{w}^*(n-1|n-1)]'G(n-1)L(n-1) \\ &\quad [P^*(n)F(n-1)\tilde{x}(n-1|n-1) + \tilde{w}^*(n-1|n-1)]. \end{aligned} \tag{2.33}$$

Here, $\tilde{w}^*(n-1|n-1)$ is a linear function of $\{\tilde{\xi}(k|k), k=N,N-1,\ldots,$

$n-1\}$; therefore, from lemma, equation (2.33) is functionally independent of the past control inputs U^{n-2}. In deriving equation (2.32), equations (2.5)-(2.12) are employed.

Now, we see from (2.32) that equation (2.31) also holds for n replaced by $n - 1$. Therefore, by induction, Theorem 5.1 holds. This completes the proof of Theorem 5.1.

From (2.4), the values of the estimates $\hat{x}(k|k)$ and $\hat{w}^*(k|k)$ are required to realize the optimal control input $u^*(k)$. By the same procedure as taken in chapter II, we can easily obtain the optimal estimates $\hat{x}(k|k)$ and $\hat{w}^*(k|k)$, though system (2.1) and (2.2) is slightly different from the system treated in chapter II.

Theorem 5.2

For system (2.1) and (2.2), the optimal estimates $\hat{x}(k|k)$ of state $x(k)$ and $\hat{w}^*(k|k)$ defined by (2.9)-(2.12) are given by the following.

$$\hat{x}(k|k) = \sum_{i_0=1}^{M} \cdots \sum_{i_k=1}^{M} \hat{x}(k|k,I^k)P(I^k|Y^k), \tag{2.34}$$

where

$$\hat{x}(k|k,I^k) \triangleq E\{x(k)|\Gamma^k=I^k,Y^k\} \qquad (2.35)_1$$

$$\Gamma^k \triangleq \{\gamma(0), \gamma(1), \ldots, \gamma(k)\} \tag{2.35$_2$}$$

and

$$I^k \triangleq \{i_0, i_1, \ldots, i_k\}. \tag{2.35$_3$}$$

Here, $\hat{x}(k|k, I^k)$ defined by (2.35) is given by

$$\hat{x}(k|k, I^k) = \hat{x}(k|k-1, I^k) + K(k, I^k)[y(k) - H(k)\hat{x}(k|k-1, I^k) - b(k, i_k)] \tag{2.36$_1$}$$

and

$$\hat{x}(k|k-1, I^k) = F(k-1)\hat{x}(k-1|k-1, I^{k-1}) + a(k-1, i_k), \tag{2.36$_2$}$$

where

$$K(k, I^k) = \hat{P}(k|k-1, I^k)H'(k)[H(k)\hat{P}(k|k-1, I^k)H'(k) + R(k, i_k)R'(k, i_k)]^{-1} \tag{2.37$_1$}$$

$$\hat{P}(k|k, I^k) = \hat{P}(k|k-1, I^k) - K(k, I^k)H(k)\hat{P}(k|k-1, I^k) \tag{2.37$_2$}$$

$$\hat{P}(k|k-1, I^k) = F(k-1)\hat{P}(k-1|k-1, I^{k-1})F'(k-1) + Q(k-1, i_k)Q'(k-1, i_k). \tag{2.37$_3$}$$

Also, the a posteriori probability $P(I^k|Y^k)$ appearing in (2.34) is given by

$$P(I^k|Y^k) = \frac{p(y(k)|I^k,Y^{k-1})P(I^k|Y^{k-1})}{\sum_{i_0=1}^{M} \cdots \sum_{i_k=1}^{M} p(y(k)|I^k,Y^{k-1})P(I^k|Y^{k-1})}, \qquad (2.38)$$

where $p(y(k)|I^k,Y^{k-1})$ is Gaussian with

$$\text{mean} = H(k)\hat{x}(k|k-1,I^k) + b(k,i_k) \qquad (2.39)_1$$

$$\text{cov} = H(k)\hat{P}(k|k-1,I^k)H'(k) + R(k,i_k)R'(k,i_k), \qquad (2.39)_2$$

and the a priori probability $P(I^k|Y^{k-1})$ appearing in (2.38) is given by

$$P(I^k|Y^{k-1}) = p_{i_{k-1},i_k}(k)P(I^{k-1}|Y^{k-1}). \qquad (2.40)$$

Equations (2.34)-(2.40) offer us the optimal estimate $\hat{x}(k|k)$ of state $x(k)$ given observations Y^k sequentially.

Next, the optimal estimate $\hat{w}^*(k|k)$ is given by

$$\hat{w}^*(k|k) = \sum_{i_{k+1}=1}^{M} \cdots \sum_{i_{N+1}=1}^{M} \hat{w}^*(k,I_{k+1}^{N+1})P(I_{k+1}^{N+1}|Y^k), \qquad (2.41)$$

where

$$I_{k+1}^{N+1} \triangleq \{i_{k+1},i_{k+2},\ldots,i_{N+1}\} \qquad (2.42)$$

and $\hat{w}^*(k,I_{k+1}^{N+1})$ is defined by

$$\hat{w}^*(k,I_{k+1}^{N+1}) = P^*(k+1)a(k,i_{k+1}) + T(k+1)\hat{w}^*(k+1,I_{k+2}^{N+1}) \qquad (2.43)_1$$

$$T(k) = F'(k) - F'(k)P^*(k+1)G(k)L(k) \qquad (2.43)_2$$

$$L(k) = [G'(k)P^*(k+1)G(k) + Q_1(k)]^{-1}G'(k) \qquad (2.43)_3$$

and

$$\hat{w}^*(N, i_{N+1}) = P(N+1)a(N, i_{N+1}). \qquad (2.43)_4$$

Also, the conditional probability $P(I_{k+1}^{N+1}|Y^k)$ is given by

$$P(I_{k+1}^{N+1}|Y^k) = \sum_{i_0=1}^{M} \cdots \sum_{i_k=1}^{M} P(I^k|Y^k)p_{i_k,i_{k+1}}(k+1)\times \cdots$$

$$\cdots \times \; p_{i_N,i_{N+1}}(N+1) \qquad (2.44)$$

and $P(I^k|Y^k)$ in (2.44) is given by equations (2.38)-(2.40).

Proof of Theorem 5.2

Equations (2.34)-(2.40) for the optimal estimate $\hat{x}(k|k)$ are obtained by the same procedure as was adopted in section 2.3, chapter II. The differences are in that bias terms $a(k-1, i_k)$ and $b(k, i_k)$ appear in equations (2.36) and (2.39). Similarly, from definition (2.9)-(2.12) of the optimal estimate $\hat{w}^*(k|k)$ and by the fact that white noise $w(k)$ is independent of observations Y^k, we can easily obtain equations (2.41)-(2.44) for the optimal estimate $\hat{w}^*(k|k)$.

Remark 1 : Theorem 5.1 also holds for linear discrete systems with arbitrary second-order noise processes such as

$$x(k+1) = F(k)x(k) + G(k)u(k) + \xi(k) \tag*{(2.45)$_1$}$$

$$y(k) = H(k)x(k) + \eta(k), \tag*{(2.45)$_2$}$$

where

$$E\{\xi'(k)\xi(k)\} < \infty$$

and

$$E\{\eta'(k)\eta(k)\} < \infty.$$

Here, it is not required that noise sequences $\{\xi(k), k=1,2,\ldots\}$ and $\{\eta(k), k=1,2,\ldots\}$ are Gaussian and/or white and that $\xi(k)$ and $\eta(k)$ are uncorrelated. The optimal control input $u^*(k)$ under the performance criterion (2.3) is given by (2.4)-(2.12) without $(2.10)_2$.*) But it should, however, be noted that for the general system (2.45) the optimal estimates $\hat{x}(k|k)$ and $\hat{w}^*(k|k)$ appearing in (2.4) can not be obtained, although for system (2.1) and (2.2) the optimal estimates are given by Theorem 5.2.

Remark 2 : As was pointed out in section 2.3, chapter II, the evergrowing amount of memory is required for the evaluation of the optimal estimate $\hat{x}(k|k)$. Therefore, the optimal control algorithm presented above becomes infeasible for large N, where N is the total length of the control interval. Thus an appro-

*) Certainty equivalence has recently been examined by Akashi and Nose (to appear).

ximate estimator algorithm should be employed in such a case. An approximate estimator algorithm for $\hat{x}(k|k)$ and $\hat{w}^*(k|k)$ can be obtained by the same approach as was taken in section 2.5.1, chapter II, as follows.

Approximate Estimator Algorithm

Approximate estimates $x^*(k|k)$ and $\overline{w}^*(k|k)$ of the optimal $\hat{x}(k|k)$ and $\hat{w}^*(k|k)$ are furnished as follows.

$$x^*(k|k) = \sum_{i=1}^{M} x_i^*(k|k)p(i|k), \tag{2.46}$$

where

$$x_i^*(k|k) = x_i^*(k|k-1) + K_i(k)[y(k) - H(k)x_i^*(k|k-1) - b(k,i)] \qquad (2.47)_1$$

$$x_i^*(k|k-1) = F(k-1)x^*(k-1|k-1) + a(k-1,i) \qquad (2.47)_2$$

$$K_i(k) = P_i^*(k|k-1)H'(k)[H(k)P_i^*(k|k-1)H'(k) + R(k,i)R'(k,i)]^{-1} \qquad (2.47)_3$$

$$P_i^*(k|k-1) = F(k-1)P^*(k-1|k-1)F'(k-1) + Q(k-1,i)Q'(k-1,i) \qquad (2.47)_4$$

$$p(i|k) = \frac{\Lambda_i(k)\sum_{j=1}^{M} p_{ji}(k)p(j|k-1)}{\sum_{i=1}^{M}\Lambda_i(k)\sum_{j=1}^{M} p_{ji}(k)p(j|k-1)} \qquad (2.47)_5$$

and $\Lambda_i(k)$ in $(2.47)_5$ is defined by

$$\Lambda_i(k) \triangleq p(y(k)|\gamma(k)=i, Y^{k-1}) \tag*{$(2.47)_6$}$$

and is Gaussian with

$$\begin{aligned} \text{mean} &= H(k)x_i^*(k|k-1) + b(k,i) \\ \text{cov} &= H(k)P_i^*(k|k-1)H'(k) + R(k,i)R'(k,i). \end{aligned} \tag*{$(2.47)_7$}$$

Also, the approximate error covariance matrix $P^*(k|k)$ is given by

$$P^*(k|k) = \sum_{i=1}^{M} \Big\{P_i^*(k|k) + [x^*(k|k) - x_i^*(k|k)] \times [x^*(k|k) - x_i^*(k|k)]'\Big\} p(i|k), \tag{2.48}$$

where

$$P_i^*(k|k) = P_i^*(k|k-1) - K_i(k)H(k)P_i^*(k|k-1). \tag{2.49}$$

Moreover, by using the approximate a posteriori probability $p(i|k) \overset{\sim}{=} P(\gamma(k)=i|Y^k)$ given by $(2.47)_5$, an approximate estimate $\overline{w}^*(k|k)$ of $w^*(k)$ given Y^k can be obtained as follows:

$$\overline{w}^*(k|k) = \sum_{i_k=1}^{M} s^*(k,i_k)p(i_k|k), \tag{2.50}$$

where

$$s^*(k,i_k) = \sum_{i_{k+1}=1}^{M} \cdots \sum_{i_{N+1}=1}^{M} \hat{w}^*(k,I_{k+1}^{N+1})p_{i_k,i_{k+1}}(k+1) \times \cdots$$

$$\cdots \times p_{i_N,i_{N+1}}(N+1) \qquad (2.51)$$

Here, $\hat{w}^*(k,I_{k+1}^{N+1})$ is defined by (2.43).

Remark 3 : Because (2.43) is a deterministic equation, $s^*(k,i_k)$ defined by (2.51) can be obtained by off-line computation. Approximate estimates $x^*(k|k)$ and $\bar{w}^*(k|k)$ given by (2.46)-(2.51) together with Theorem 5.1 furnish a suboptimal control input for system (2.1) and (2.2) with criterion (2.3).

5.3 Optimal Control for Continuous-Discrete System with Switching Environment

In this section, we shall consider the stochastic control problem for a class of linear continuous-discrete systems with switching environments, where the characteristics of the system- and the measurement-noise processes change according to a jump Markov process. We shall show that a certainty equivalence (Patchel and Jacobs, 1971) holds for linear continuous-discrete systems with switching environments.

Consider the system described by a stochastic differential equation:

$$dx(t) = F(t)x(t)dt + G(t)u(t)dt + a(t,\gamma(t))dt$$
$$+ Q(t,\gamma(t))dw(t), \quad 0 \leqq t \leqq T, \qquad (3.1)_1$$

and let the observation be given by

$$y(t_k) = H(t_k)x(t_k) + b(t_k,\gamma(t_k)) + R(t_k,\gamma(t_k))v(t_k),$$
$$0 < t_1 < t_2 < \cdots < t_N < T, \qquad (3.1)_2$$

where

$x(t)$: an $n \times 1$ state vector;

$y(t_k)$: a $p \times 1$ observation vector;

$u(t)$: an $r \times 1$ control input vector;

$F(t)$: an $n \times n$ state transition matrix;

$G(t)$: an $n \times r$ control gain matrix;

$H(t_k)$: a $p \times n$ observation matrix;

$a(t,\cdot)$: an $n \times 1$ vector;

$b(t_k,\cdot)$: a $p \times 1$ vector;

$Q(t,\cdot)$: an $n \times m$ matrix;

$R(t_k,\cdot)$: a $p \times p$ nonsingular matrix;

$w(t)$: an $m \times 1$ standard Wiener process;

$v(t_k)$: a $p \times 1$ zero-mean Gaussian white noise sequence with unit variance matrix;

and

$\gamma(t)$: a right-continuous jump Markov process taking on values of 1, 2,..., M with transition rates

$$q_{ij}(t) = \lim_{s \downarrow t} \frac{P(\gamma(s)=j \mid \gamma(t)=i)}{s - t}, \tag{3.2$_1$}$$

$i, j = 1, 2, \ldots, M;\quad i \neq j.$

Also, define:

$$q_i(t) = \sum_{\substack{j=1 \\ j \neq i}}^{M} q_{ij}(t). \tag{3.2$_2$}$$

It is assumed that the control input $u(t)$ is Y^{t_k}-measurable for $t_k \leqq t < t_{k+1}$, where Y^{t_k} is a minimum σ-field generated by $\{y(t_1), y(t_2), \ldots, y(t_k)\}$. That is to say, the control input $u(t)$ for $t_k \leqq t < t_{k+1}$ is determined based upon the observations $\{y(t_1), y(t_2), \ldots, y(t_k)\}$. It is also assumed that stochastic processes $w(t)$, $v(t_k)$, $\gamma(t)$ and $x(0)$ are mutually independent and that $x(0)$ is Gaussian with $E\{x(0)\} = \hat{x}(0)$ and $Cov\{x(0)\} = \hat{P}(0)$.

The objective of this section is to find the optimal control policy which minimizes the following expected quadratic cost:

$$J = E\left\{\|x(T)\|^2_{P_T} + \int_0^T \|x(t)\|^2_{P(t)} dt + \int_0^T \|u(t)\|^2_{Q_1(t)} dt\right\}, \tag{3.3}$$

where P_T and $P(t)$ are $n \times n$ symmetric nonnegative definite matrices and $Q_1(t)$ an $r \times r$ symmetric positive definite matrix.

The optimal control policy can be found by the similar approach as was taken in the previous section. The following theorem completely specifies the optimal control algorithm.

Theorem 5.3

The optimal control input $u^*(t)$ for system (3.1) with criterion (3.3) is given by

$$u^*(t) = - K(t)\hat{x}(t|t_k) - L(t)\hat{w}^*(t|t_k) \tag{3.4}$$

for $t_k \leqq t < t_{k+1}$, where

$$K(t) = Q_1^{-1}(t)G'(t)P^*(t) \tag*{$(3.5)_1$}$$

$$L(t) = Q_1^{-1}(t)G'(t) \tag*{$(3.5)_2$}$$

$$\dot{P}^*(s) + P^*(s)F(s) + F'(s)P^*(s) + P(s) - P^*(s)G(s)Q_1^{-1}(s)G'(s)P^*(s) = 0, \tag*{$(3.5)_3$}$$

$$\text{for} \quad 0 \leqq s \leqq T,$$

$$P^*(T) = P_T, \tag*{$(3.5)_4$}$$

and

$$\hat{x}(t|t_k) \triangleq E\{x(t)|Y^{t_k}\}$$

$$\dot{\hat{w}}^*(s|t_k) + F'(s)\hat{w}^*(s|t_k) + P^*(s)\hat{a}(s,t_k)$$

$$- P^*(s)G(s)Q_1^{-1}(s)G'(s)\hat{w}^*(s|t_k) = 0,$$

$$\text{for} \quad t_k \leq s \leq t_{k+1}, \quad k = N, N-1, \ldots, 0, \qquad (3.6)_1$$

$$\hat{w}^*(t_{k+1}|t_k) = E\{w^*(t_{k+1}|t_{k+1})|Y^{t_k}\}, \quad k = 0, 1, \ldots, N-1, \qquad (3.6)_2$$

$$\hat{w}^*(T|t_N) = 0, \qquad (3.6)_3$$

$$\hat{a}(s,t_k) \triangleq E\{a(s,\gamma(s))|Y^{t_k}\}.$$

Here, the upper dot denotes differentiation with respect to time s, and we set $t_0 = 0$ and $t_{N+1} = T$.

Proof of Theorem 5.3

Let us consider processes $y_1(t_k)$ and $y_2(t_k)$ generated by

$$dx_1(t) = F(t)x_1(t)dt + a(t,\gamma(t))dt + Q(t,\gamma(t))dw(t),$$

$$x_1(0) = x(0), \qquad (3.7)_1$$

$$y_1(t_k) = H(t_k)x_1(t_k) + b(t_k,\gamma(t_k)) + R(t_k,\gamma(t_k))v(t_k), \qquad (3.7)_2$$

and

$$dx_2(t) = F(t)x_2(t)dt + G(t)u(t)dt, \quad x_2(0) = 0, \qquad (3.8)_1$$

$$y_2(t_k) = H(t_k)x_2(t_k), \tag{3.8$_2$}$$

respectively. From equations (3.1), (3.7) and (3.8), we have

$$x(t) = x_1(t) + x_2(t) \tag{3.9}$$

and

$$y(t_k) = y_1(t_k) + y_2(t_k). \tag{3.10}$$

Here, $x_1(t)$ and $y_1(t_k)$ are independent of the control input; while $x_2(t)$ and $y_2(t_k)$ are completely determined by the control input. Therefore, for a random variable z independent of the control input, we have

$$E\{z|U^t, Y^{t_k}\} = E\{z|Y_1^{t_k}\}, \tag{3.11}$$

where $U^t \triangleq \{u(s), 0 \leq s \leq t\}$, $Y_1^{t_k} \triangleq \{y_1(t_1), y_1(t_2), \ldots, y_1(t_k)\}$; and equation (3.11) is functionally independent of control inputs U^t. Thus, from (3.7)-(3.11), we can easily show the following lemma.

<u>lemma</u> : Following variables:

$$\tilde{x}(t|t_k) \triangleq x(t) - \hat{x}(t|t_k) \;\; (= x(t) - E\{x(t)|U^t, Y^{t_k}\})$$

and

$$\hat{a}(t,t_k) \triangleq E\{a(t,\gamma(t))|U^t, Y^{t_k}\}, \quad t_k \leq t$$

are functionally independent of the past control inputs U^t.

Proof of lemma : From (3.7)-(3.11),

$$\tilde{x}(t|t_k) = x_1(t) - E\{x_1(t)|U^t, Y^{t_k}\}$$
$$= x_1(t) - E\{x_1(t)|Y_1^{t_k}\}: \text{ independent of } U^t.$$

Also, since $\gamma(t)$ is functionally independent of past control inputs U^t, we see from (3.11) that $\hat{a}(t,t_k)$ is functionally independent of past control inputs U^t.

Based upon the above lemma, we shall derive the optimal algorithm. Let us define:

$$J(t_k) \triangleq E\left\{\|x(T)\|^2_{P_T} + \int_{t_k}^{T} \|x(t)\|^2_{P(t)}dt + \int_{t_k}^{T} \|u(t)\|^2_{Q_1(t)}dt \,|\, U^{t_k}, Y^{t_k}\right\} \tag{3.12}$$

and

$$J^*(t_k) \triangleq \min_{\{u(s),\, t_k \leq s \leq T\}} J(t_k). \tag{3.13}$$

Then, by Bellman's principle of optimality (Bellman, 1957), the optimal control $u^*(s)$ $(t_k \leq s < t_{k+1})$ is given by $u(s)$ which minimizes the right-hand side of the following functional equation:

$$J^*(t_k) = \min_{\{u(s), t_k \leq s < t_{k+1}\}} E\left\{\int_{t_k}^{t_{k+1}} \|x(s)\|^2_{P(s)} ds + \int_{t_k}^{t_{k+1}} \|u(s)\|^2_{Q_1(s)} ds + J^*(t_{k+1}) \,|\, U^{t_{k+1}}, Y^{t_k}\right\} \tag{3.14}$$

with $t_{N+1} = T$ and

$$J^*(T) = \|x(T)\|^2_{P_T} .$$

For $k = N$, we have from (3.12)

$$J(t_N) = \|\hat{x}(T|t_N)\|^2_{P_T} + \int_{t_N}^{T} \|\hat{x}(t|t_N)\|^2_{P(t)} dt + \int_{t_N}^{T} \|u(t)\|^2_{Q_1(t)} dt + I_0(t_N), \tag{3.15}$$

where $I_0(t_N)$ is a functional of estimation error $\tilde{x}(t|t_N)$ ($t_N \leq t \leq T$) and is functionally independent of $\{u(t), 0 \leq t \leq T\}$, from lemma. Also, from (3.1), we have

$$\dot{\hat{x}}(t|t_N) = F(t)\hat{x}(t|t_N) + G(t)u(t) + \hat{a}(t,t_N), \tag{3.16}$$

$$t_N \leq t \leq T.$$

Therefore, from (3.15) and (3.16), we see that the problem of the minimization of $J(t_N)$ with respect to $\{u(s), t_N \leq s \leq T\}$ is merely the usual linear deterministic control problem. Performing the minimization of (3.14) for $k = N$ yields the optimal control inputs $u^*(t)$ $(t_N \leq t \leq T)$ which is given by (3.4)-(3.6) for $k = N$. As a result, equation (3.14) becomes

$$J^*(t_N) = \|\hat{x}(t_N|t_N)\|^2_{P^*(t_N)} + 2\hat{x}'(t_N|t_N)\hat{w}^*(t_N|t_N) + I_1(t_N), \tag{3.17}$$

where $I_1(t_N)$ is a functional of $\{\tilde{x}(t|t_N), \hat{a}(t,t_N), t_N \leq t \leq T\}$ and is functionally independent of the past control policy, from lemma.

From (3.17), we have

$$E\{J^*(t_N)|U^{t_N}, Y^{t_{N-1}}\} = \|\hat{x}(t_N|t_{N-1})\|^2_{P^*(t_N)} + 2\hat{x}'(t_N|t_{N-1})\hat{w}^*(t_N|t_{N-1}) + I_2(t_{N-1}), \tag{3.18}$$

where $\hat{w}^*(t_N|t_{N-1}) \triangleq E\{\hat{w}^*(t_N|t_N)|Y_1^{t_{N-1}}\}$ and $I_2(t_{N-1})$ is a functional of $\{\tilde{x}(t_N|t_N), \tilde{x}(t|t_{N-1}), \hat{a}(t,t_{N-1}), t_{N-1} \leq t \leq t_N\}$ and is independent of the past control inputs U^{t_N}. Also, we can obtain the following equation which is similar to (3.16):

$$\dot{\hat{x}}(t|t_{N-1}) = F(t)\hat{x}(t|t_{N-1}) + G(t)u(t) + \hat{a}(t,t_{N-1}) \tag{3.19}$$

for $t_{N-1} \leq t \leq t_N$. Moreover, from (3.18), equation (3.14) for $k = N-1$ becomes

$$J^*(t_{N-1}) = \min_{\{u(t), t_{N-1} \leq t < t_N\}} \left\{ \int_{t_{N-1}}^{t_N} \|\hat{x}(t|t_{N-1})\|^2_{P(t)} dt + \right.$$

$$+ \int_{t_{N-1}}^{t_N} \|u(t)\|^2_{Q_1(t)} dt + \|\hat{x}(t_N|t_{N-1})\|^2_{P*(t_N)}$$

$$+ 2\hat{x}'(t_N|t_{N-1})\hat{w}*(t_N|t_{N-1}) + \overline{I}_2(t_{N-1}) \ , \qquad (3.20)$$

where $\overline{I}_2(t_{N-1})$ is a functional of $\{\tilde{x}(t_N|t_N), \tilde{x}(t|t_{N-1}), \hat{a}(t,t_{N-1}), t_{N-1} \leqq t \leqq t_N\}$ and is functionally independent of the past control policy. Therefore, from (3.19) and (3.20) we can easily perform the minimization of (3.20) and the optimal control input $u*(t)$ $(t_{N-1} \leqq t < t_N)$ is given by (3.4)-(3.6) for $k = N-1$, which results in

$$J*(t_{N-1}) = \|\hat{x}(t_{N-1}|t_{N-1})\|^2_{P*(t_{N-1})} + 2\hat{x}'(t_{N-1}|t_{N-1})\hat{w}*(t_{N-1}|t_{N-1})$$

$$+ I_3(t_{N-1}), \qquad (3.21)$$

where $I_3(t_{N-1})$ is a term which is functionally independent of the past control policy. Thus, from (3.17) and (3.21), we can show by induction that the optimal control input $u*(t)$ $(0 \leqq t \leqq T)$ is given by (3.4)-(3.6). The details are omitted here.

Remark 1 : If for fixed t_k we define $\overline{w}(s|t_k)$ process $(t_k \leqq s \leqq T)$ by

$$\dot{\overline{w}}(s|t_k) + F'(s)\overline{w}(s|t_k) + P*(s)\hat{a}(s,t_k)$$

$$- P*(s)G(s)Q_1^{-1}(s)G'(s)\overline{w}(s|t_k) = 0,$$

$$t_k \leqq s \leqq T \qquad (3.22)$$

with $\overline{w}(T|t_k) = 0$, then there holds

$$w^*(s|t_k) = \overline{w}(s|t_k), \quad \text{for} \quad t_k \leqq s \leqq t_{k+1}, \tag{3.23}$$

where $\hat{w}(s|t_k)$ $(t_k \leqq s \leqq t_{k+1})$ is defined by (3.6).

Remark 2 : It is to be noted that the first term in the right-hand side of (3.4) corresponds to the optimal control input for usual linear-quadratic-Gaussian systems and that the second one is the correction term due to the fact that the noise processes are non-white and have non-zero means.

Remark 3 : Theorem 5.3 shows that a certainty equivalence (Patchel and Jacobs, 1971) holds for system (3.1) with criterion (3.3). It is to be noted that the certainty equivalence also holds for the following system (Fujishige and Sawaragi, submitted 1974):

$$dx(t) = F(t)x(t)dt + G(t)u(t)dt + d\overline{w}(t) \qquad (3.24)_1$$

$$y(t_k) = H(t_k)x(t_k) + \overline{v}(t_k), \qquad (3.24)_2$$

where $\{\overline{v}(t_k),\ k=0,1,\ldots\}$ is an arbitrary second-order noise sequence and $\{\overline{w}(t), 0 \leqq t \leqq T\}$ is also an arbitrary second-order noise process with $\overline{w}(0) = 0$ which satisfies

$$\sup_{0 \leqq t \leqq T} E\{\overline{w}'(t)\overline{w}(t)\} < \infty$$

and

$$E\{\overline{w}(u)|U^t, Y^{t_k}\} = \int_0^u \hat{a}(s,t_k)ds$$

for $u \geqq t \geqq t_k$. Here, $\hat{a}(s,t_k)$ is assumed to be independent of the control policy. Under the criterion (3.3), the optimal control input $u^*(t)$ for system (3.24) is given by the same equations (3.4)-(3.6). System (3.24) includes system (3.1) as a special case.

Now, we shall show the optimal estimator algorithm for $\hat{x}(t|t_k)$ and $\hat{w}^*(t|t_k)$ for $t_k \leqq t \leqq t_{k+1}$, $k = 0, 1, \ldots, N$, which are required for us to realize the optimal control input $u^*(t)$ in (3.4).

Theorem 5.4

The optimal estimate $\hat{x}(t|t_k)$ $(t_k \leqq t \leqq t_{k+1})$ is given by

$$\hat{x}(t|t_k) = \sum_{i_0=1}^{M} \hat{x}(t|t_k, \{\gamma(0)=i_0, \tau_1>t\}) P(\gamma(0)=i_0, \tau_1>t|Y^{t_k})$$

$$+ \sum_{n=1}^{\infty} \sum_{i_0=1}^{M} \sum_{\substack{i_1=1 \\ i_1 \neq i_0}}^{M} \cdots \sum_{\substack{i_n=1 \\ i_n \neq i_{n-1}}}^{M} \int_0^t \int_{s_1}^t \cdots \int_{s_{n-1}}^t \hat{x}(s|t_k, \{\gamma(0)=i_0, \tau^n=s^n, j^n=i^n, \tau_{n+1}>t\})$$

$$\times P(\gamma(0)=i_0, \tau^n \varepsilon ds^n, j^n=i^n, \tau_{n+1}>t|Y^{t_k}), \qquad (3.25)$$

where the a posteriori probabilities $P(\gamma(0)=i_0, \tau_1>t|Y^{t_k})$ and $P(\gamma(0)=i_0, \tau^n \varepsilon ds^n, j^n=i^n, \tau_{n+1}>t|Y^{t_k})$ are, respectively, given by

$$P(\gamma(0)=i_0,\tau_1>t|Y^{t_k}) = \frac{p(y(t_k)|\gamma(0)=i_0,\tau_1>t,Y^{t_{k-1}})P(\gamma(0)=i_0,\tau_1>t|Y^{t_{k-1}})}{p(y(t_k)|Y^{t_{k-1}})} \qquad (3.26)_1$$

and

$$P(\gamma(0)=i_0,\tau^n \varepsilon ds^n, j^n=i^n, \tau_{n+1}>t|Y^{t_k})$$

$$= \frac{p(y(t_k)|\gamma(0)=i_0,\tau^n=s^n,j^n=i^n,\tau_{n+1}>t,Y^{t_{k-1}})}{p(y(t_k)|Y^{t_{k-1}})} *$$

$$* \frac{P(\gamma(0)=i_0,\tau^n \varepsilon ds^n, j^n=i^n, \tau_{n+1}>t|Y^{t_{k-1}})}{} . \qquad (3.26)_2$$

Here, $p(y(t_k)|\gamma(s),0\leqq s\leqq t,Y^{t_{k-1}})$ is Gaussian with

$$\text{mean} = H(t_k)\hat{x}(t_k|t_{k-1},\{\gamma(s),0\leqq s\leqq t_k\}) + b(t_k,\gamma(t_k)) \qquad (3.27)_1$$

$$\text{cov} = H(t_k)\hat{P}(t_k|t_{k-1},\{\gamma(s),0\leqq s\leqq t_k\})H'(t_k)$$

$$+ R(t_k,\gamma(t_k))R'(t_k,\gamma(t_k)), \qquad (3.27)_2$$

where

$$\hat{x}(t|t_k,\{\gamma(s),0\leqq s\leqq t\}) \triangleq E\{x(t)|\gamma(s),0\leqq s\leqq t,Y^{t_k}\} \qquad (3.28)$$

and

$$\hat{P}(t|t_k,\{\gamma(s),0\leqq s\leqq t\}) \triangleq Cov\{x(t)|\gamma(s),0\leqq s\leqq t,Y^{t_k}\}. \qquad (3.29)$$

Moreover, the conditional probability density function $p(y(t_k)|Y^{t_{k-1}})$ appearing in (3.26) is given by

$$p(y(t_k)|Y^{t_{k-1}})$$

$$= \sum_{i_0=1}^{M} p(y(t_k)|\gamma(0)=i_0,\tau_1>t_k,Y^{t_{k-1}})P(\gamma(0)=i_0,\tau_1>t_k|Y^{t_{k-1}})$$

$$+ \sum_{n=1}^{\infty} \sum_{i_0=1}^{M} \sum_{\substack{i_1=1\\ i_1\neq i_0}}^{M} \cdots \sum_{\substack{i_n=1\\ i_n\neq i_{n-1}}}^{M} \int_0^{t_k}\int_{s_1}^{t_k}\cdots\int_{s_{n-1}}^{t_k}$$

$$p(y(t_k)|\gamma(0)=i_0,\tau^n=s^n,j^n=i^n,\tau_{n+1}>t_k,Y^{t_{k-1}})$$

$$\times P(\gamma(0)=i_0,\tau^n\varepsilon ds^n,j^n=i^n,\tau_{n+1}>t_k|Y^{t_{k-1}}). \qquad (3.30)$$

Furthermore, $\hat{w}^*(t|t_k)$ $(t_k \leq t \leq t_{k+1})$ is given by equations $(3.6)_1$-$(3.6)_3$ or by equation (3.22), where $\hat{a}(s,t_k) \triangleq E\{a(s,\gamma(s))|Y^{t_k}\}$ is obtained by

$$\hat{a}(s,t_k) = \sum_{i=1}^{M} a(s,i)P(\gamma(s)=i|Y^{t_k}) \qquad (3.31)$$

and

$$P(\gamma(s)=i|Y^{t_k}) = P(\gamma(0)=i,\tau_1>s|Y^{t_k})$$

$$+ \sum_{n=1}^{\infty} \sum_{i_0=1}^{M} \sum_{\substack{i_1=1\\ i_1\neq i_0}}^{M} \cdots \sum_{\substack{i_{n-1}=1\\ i_{n-1}\neq\{i_{n-2},i\}}}^{M} \int_0^{s}\int_{u_1}^{s}\cdots\int_{u_{n-1}}^{s}$$

$$P(\gamma(0)=i_0,\tau^n\varepsilon du^n,j^{n-1}=i^{n-1},j_n=i,\tau_{n+1}>s|Y^{t_k}). \qquad (3.32)$$

Here, the notation adopted follows the one in chapter IV.

Remark 4 : We can also express $\hat{a}(s,t_k)$ as follows:

$$\hat{a}(s,t_k) = \sum_{i=1}^{M} \alpha_k(s,i) P(\gamma(t_k)=i \mid Y^{t_k}), \tag{3.33}$$

where

$$\alpha_k(s,i) \triangleq E\{a(s,\gamma(s)) \mid \gamma(t_k)=i\}, \quad s \geqq t_k. \tag{3.34}$$

Equations (3.33) and (3.34) may be more convenient than equations (3.31) and (3.32) for on-line computation, because $\alpha_k(s,i)$ is a deterministic function of k, s and i, and can be obtained a priori for $t_k \leqq s \leqq T$, $k = 1, 2, \ldots, N$ and $i = 1, 2, \ldots, M$.

Remark 5 : Since the optimal estimator algorithms for $\hat{x}(t|t_k)$ and $\hat{w}^*(t|t_k)$ are not feasible, approximate estimator algorithms are required for the practical implementation. We can obtain approximate estimator algorithms by the similar procedure as was taken in chapter IV.

5.4 Suboptimal Control for Discrete System with Markov Chain

We shall consider, in this section, a stochastic optimal control problem for a class of linear discrete systems with a Markov chain. In section 5.2, we have already considered the stochastic optimal control problem for linear discrete systems including a Markov chain in a special way, that is, linear discrete systems with switching environments. More general discrete systems will be treated in the following.

Let us consider the system represented by a stochastic difference equation:

$$x(k+1) = F(k,\gamma(k+1))x(k) + G(k,\gamma(k+1))u(k) + Q(k,\gamma(k+1))w(k), \tag{4.1$_1$}$$

and let the observation be given by

$$y(k) = H(k,\gamma(k))x(k) + R(k,\gamma(k))v(k), \tag{4.1$_2$}$$

where the notation and the assumption are the same as in section 5.2. The difference between the present system (4.1) and system (2.1) and (2.2) is that matrices F, G and H in (4.1) depend upon the Markov chain $\gamma(k)$ and that system (4.1) does not contain random-bias terms.

The problem is to find the optimal control policy which minimizes the performance criterion given by (2.3) in section 5.2. However, the optimal control policy cannot be found for system (4.1) with criterion (2.3) (cf. Fujishige, 1974); therefore, a suboptimal control algorithm will be presented in the sequel.

Let us define:

$$J(k) = E\left\{ \sum_{i=k}^{N} [x'(k+1)P(k+1)x(k+1) + u'(k)Q_1(k)u(k)] \,|\, U^k, Y^k \right\}, \qquad (4.2)$$

and

$$J^*(k) = \min_{\{u(i),\, k \leq i \leq N\}} J(k). \qquad (4.3)$$

Then, by Bellman's principle of optimality (Bellman, 1957),

$$J^*(k) = \min_{u(k)} E\left\{ x'(k+1)P(k+1)x(k+1) + u'(k)Q_1(k)u(k) + J^*(k+1) \,|\, U^k, Y^k \right\}, \quad k = N, N-1, \ldots, 0, \qquad (4.4)$$

and

$$J^*(N+1) \equiv 0.$$

The above functional equation cannot be soved analytically; therefore, in order to obtain a near-optimal control policy, we approximate $J^*(k+1)$ in (4.4) by assuming as follows.

Assumption : The true values of $x(i)$ and $\gamma(i)$ are given at each time i for $k \leq i \leq N$.

It is to be noted that the true values of future $x(i)$ and $\gamma(i)$ $(k<i\leq N)$ are not given at time k.

Based upon Assumption (4.5), $J^*(k+1)$ in (4.4) is given by

$$J^*(\ell,x(\ell),i) = \min_{u(\ell)} E\Big\{x'(\ell+1)P(\ell+1)x(\ell+1) + u'(\ell)Q_1(\ell)u(\ell) + J^*(\ell+1,x(\ell+1),\gamma(\ell+2)) \,\Big|\, x(\ell),\gamma(\ell+1)=i\Big\} \qquad (4.6)_1$$

for $\ell = N, N-1, \ldots, k+1$, with

$$J^*(N+1,x(N+1),\gamma(N+2)) \equiv 0. \qquad (4.6)_2$$

Suppose that

$$J^*(\ell+1,x(\ell+1),\gamma(\ell+2)) = x'(\ell+1)S(\ell+1,\gamma(\ell+2))x(\ell+1) + I(\ell+1,\gamma(\ell+2)), \qquad (4.7)$$

where $I(\ell+1,\gamma(\ell+2))$ is functionally independent of the control policy. Substituting (4.7) into $(4.6)_1$ yields

$$\begin{aligned} &J^*(\ell,x(\ell),i) \\ &= \min_{u(\ell)} \Big\{[F(\ell,i)x(\ell) + G(\ell,i)u(\ell)]'[P(\ell+1) + T(\ell+1,i)] \\ &\qquad \times [F(\ell,i)x(\ell) + G(\ell,i)u(\ell)] \\ &\qquad + u'(\ell)Q_1(\ell)u(\ell) + I(\ell,i)\Big\}, \end{aligned} \qquad (4.8)$$

where

$$T(\ell+1,i) = \sum_{j=1}^{M} S(\ell+1,j)p_{ij}(\ell+2) \qquad (4.9)$$

and

$$I(\ell,i) = \sum_{j=1}^{M} I(\ell+1,j)p_{ij}(\ell+2)$$
$$+ \operatorname{tr} Q'(\ell,i)[P(\ell+1) + T(\ell+1,i)]Q(\ell,i). \tag{4.10}$$

Minimizing (4.8) with respect to $u(\ell)$ yields

$$J^*(\ell,x(\ell),i) = x'(\ell)S(\ell,i)x(\ell) + I(\ell,i), \tag{4.11}$$

where

$$S(\ell,i) = F'(\ell,i)P^*(\ell+1,i)F(\ell,i) - P_0(\ell,i) \qquad (4.12)_1$$

$$P^*(\ell+1,i) = P(\ell+1) + T(\ell+1,i) \qquad (4.12)_2$$

and

$$P_0(\ell,i) = F'(\ell,i)P^*(\ell+1,i)G(\ell,i)$$
$$\times [G'(\ell,i)P^*(\ell+1,i)G(\ell,i) + Q_1(\ell)]^{-1}$$
$$\times G'(\ell,i)P^*(\ell+1,i)F(\ell,i). \qquad (4.12)_3$$

Thus equation (4.7) also holds for ℓ replaced by $\ell-1$. Also, for $\ell = N+1$, equation (4.11) holds with

$$S(N+1,i) \equiv 0 \qquad (4.13)_1$$

and

$$I(N+1,i) \equiv 0 \qquad (4.13)_2$$

from $(4.6)_2$. Therefore, by induction, equation (4.11) holds for

$= N+1, N, \ldots,$ and $J^*(k+1, x(k+1), \gamma(k+2))$ is given by

$$J^*(k+1, x(k+1), \gamma(k+2))$$
$$= x'(k+1)S(k+1, \gamma(k+2))x(k+1) + I(k+1, \gamma(k+2)). \qquad (4.14)$$

Replacing $J^*(k+1)$ in (4.4) by $J^*(k+1, x(k+1), \gamma(k+2))$ in (4.14), we have

$$\begin{aligned} J^*(k) = \min_{u(k)} E\Big\{ & [F(k,\gamma(k+1))x(k) + G(k,\gamma(k+1))u(k)]' \\ & \times P^*(k+1,\gamma(k+1)) \\ & \times [F(k,\gamma(k+1))x(k) + G(k,\gamma(k+1))u(k)] \\ & + u'(k)Q_1(k)u(k) + I(k,\gamma(k+1)) \,|\, U^k, Y^k \Big\}. \end{aligned} \qquad (4.15)$$

Performing the minimization of (4.15) with respect to $u(k)$, we have a suboptimal control input:

$$u^*(k) = - \hat{A}^{-1}(k)\hat{b}(k), \qquad (4.16)$$

where

$$\hat{A}(k) = E\{G'(k,\gamma(k+1))P^*(k+1,\gamma(k+1))G(k,\gamma(k+1)) + Q_1(k) \,|\, Y^k\} \qquad (4.17)$$

and

$$\hat{b}(k) = E\{G'(k,\gamma(k+1))P^*(k+1,\gamma(k+1))F(k,\gamma(k+1))x(k) \,|\, Y^k\}. \qquad (4.18)$$

We thus have a suboptimal control algorithm as follows.

Suboptimal Control Algorithm

A suboptimal control input $u^*(k)$ for system (4.1) with criterion (2.3) is given by

$$u^*(k) = -\hat{A}^{-1}(k)\hat{b}(k), \tag{4.19}$$

where $\hat{A}(k)$ and $\hat{b}(k)$ are defined by (4.17) and (4.18), respectively. In (4.17) and (4.18), matrices $P^*(k+1,i)$ $(i=1,2,\ldots,M)$ are given by (4.9)-(4.12) together with $(4.13)_1$.

Remark : Here, the conditional estimates $\hat{A}(k)$ and $\hat{b}(k)$ are necessary for us to obtain (4.19). By the application of the approximate estimator algorithm presented in section 2.5.1, chapter II, $\hat{A}(k)$ and $\hat{b}(k)$ are approximately given by

$$\hat{A}(k) = Q_1(k) + \sum_{i=1}^{M}\sum_{j=1}^{M} G'(k,j)P^*(k+1,j)G(k,j)p_{ij}(k+1)p(i|k) \tag{4.20}$$

and

$$\hat{b}(k) = \sum_{i=1}^{M}\sum_{j=1}^{M} G'(k,j)P^*(k+1,j)F(k,j)x^*_i(k|k)p_{ij}(k+1)p(i|k), \tag{4.21}$$

where $p(i|k)$ and $x^*_i(k|k)$ are given by (5.2)-(5.10) in section 2.5.1, chapter II.

5.5 Concluding Remarks

We considered the optimal control problems for linear stochastic systems with jump parameters. It was shown that a certainty equivalence holds for both linear discrete systems and continuous-discrete systems with switching environments. As was pointed out in sections 5.2 and 5.3, the certainty equivalence also holds for linear systems with general second-order noise processes. However, the optimal control algorithm for linear switching environment systems is not feasible, so that combining the feasible approximate estimator algorithms proposed in the previous chapters, feasible suboptimal control algorithm was also presented. Furthermore, a suboptimal control algorithm was proposed for linear discrete systems modulated by a Markov chain, for which the optimal control policy cannot be obtained analytically. By applying the approximate estimator algorithm presented in section 2.5.1, chapter II, we have a feasible near-optimal control algorithm.

The optimal control problem for linear continuous systems with jump parameters was not considered here; but we can obtain a feasible suboptimal control algorithm by combining the approximate estimator algorithm proposed in section 3.4, chapter III and the optimal controller due to Wonham (1970) and Sworder (1969), where considered was the case when the complete information on both the state and the jump parameters was available to the controller. Related optimal control problems of non-Gaussian systems have recently been treated for discrete systems by Alspach and Sorenson (1971) and for continuous systems by Kwakernaak (1974).

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